Math 3510 Final Exam

This final exam covers the entire course content, from the beginning of the semester onwards. Please work carefully and check your answers where possible. It's been a pleasure teaching this class, and I hope to see you all next year! Please complete the following questions, **writing no** *more* **than one question per page.** Be sure to write your name on the front page *only,* to number your pages, and to *staple your exam when you hand it in.*

1. Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$, and $x + y = z = 1$.

2. A coordinate system called "elliptical polar coordinates" is defined by the equations $x = ar\cos\theta$, $y = br\sin\theta$, where a and b are constants. Please use the change of variables theorem to write down the integral for the area of the "elliptical annulus" $r \in [1,2]$, $\theta \in [0, 2\pi]$ in the $x - y$ plane.

3. Find the center of mass of a hemisphere in $\mathbb{R}^3$.

4. Pull back the 2-form $\omega = (-z\,dx + y\,dz) \wedge (x^2\,dy)$ from $\mathbb{R}^3$ to $\mathbb{R}^2$ under the map

$$(u, v) \to (u^2 v, \sin v \cos u, e^{uv}).$$

5. Use Green's theorem to find the area inside the horrible looking curve $x^{2/3} + y^{2/3} = 1$ by integration over the curve. (Hint: You might want to parametrize the curve by $g(t) = (\cos^3 t, \sin^3 t)$.

6. Consider the "Hershey's kiss" defined by the equations $r = 1 - z^2$ for $z \in [0, -1]$ in cylindrical coordinates in $\mathbb{R}^3$. Compute the flux of the vector field

$$V(x, y, z) = (yz, x^3, y^2) \tag{1}$$

   over this surface. (Hint: This is a trick question.)

7. Use Stokes' Theorem to prove the following: There is no smooth function $r : D^3 \to S^2$ with the property that $r(x) = x$ on S^2. (D^3 is the set $(x, y, z) \in \mathbb{R}^3$ with $x^2 + y^2 + z^2 \leq 1$.) (Hint: The integral of the area form ω on S^2 is 4π. Now pullback ω to another copy of S^2...)

8. Let A be the matrix

$$A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}. \tag{2}$$

   Compute A^{12}.

9. Compute the dimensions of the homology groups of a solid cube with two (disjoint) inner cubes hollowed out. (Hint: These were the "strange vector spaces" we covered in the last week of class.)