MATH 3500
THIRD MIDTERM

1. Rules

The third midterm is due by class on Monday, November 25. *There will be no exceptions to this policy* (it's unfair to the other students to allow anyone extra time). You are welcome to turn in the exam early if you wish.

You are permitted to use any mechanical or electronic devices that you feel will assist you on the exam. However, please read the questions carefully: if you are required to show your work (e.g. a sequence of row operations leading to a row-reduced matrix) then it is not sufficient to merely state the row-reduced form; you must show all the intermediate matrices, as well.

You are not permitted to consult other people in your work on the exam. (In particular, Dr. Shifrin has been informed of this.) Please write up your work carefully, with no more than one question per page of notebook paper.

Good luck!

2. Questions

(1) Find a basis for the space of solutions to the set of linear equations

$$\begin{aligned} 2x + 3y + z &= 0 \\ x + y - z &= 0 \\ 3x + 4y &= 0 \\ 5x + y + z &= 0 \end{aligned}$$

Once you have done so, find a basis for the set of column vectors $[b_1 b_2 b_3 b_4]^T$ for which there exist x, y, and z with

$$\begin{aligned} 2x + 3y + z &= b_1 \\ x + y - z &= b_2 \\ 3x + 4y &= b_3 \\ 5x + y + z &= b_4. \end{aligned}$$

(2) Find the inverse of the matrix below, or prove that it does not exist.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 & 17 & 18 \\ 18 & 17 & 16 & 15 & 14 & 13 \\ 12 & 11 & 10 & 9 & 8 & 7 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix}$$

If you choose to calculate the inverse directly, you may wish to use a calculator or computer to help with the arithmetic, or to check your work. This is encouraged. But I would like to see your row operations.

(3) Find bases and state the dimension of each of the following vector spaces:
   (a) 2×2 matrices.
   (b) *symmetric* 2×2 matrices.
   (c) *upper triangular* 2×2 matrices.

You must *prove* that your set of vectors is a basis for each vector space.

(4) Suppose A is an $n \times k$ matrix and B is a $k \times m$ matrix. Prove that

$$\operatorname{rank} AB \leq \operatorname{rank} A \quad \text{and} \quad \operatorname{rank} AB \leq \operatorname{rank} B.$$

(5) Prove Proposition 3.9 in your book.

(6) Suppose that V is a collection of vectors in R^n which is *not* neccesarily a subspace of R^n. We can still define $V^\perp$ to be the space of vectors $w \in R^n$ such that $w \cdot v = 0$ for each $v \in V$.

Prove or disprove: $(V^\perp)^\perp$ is the smallest subspace of R^n containing V.