Asymptotic Analysis of Discrete Normals and Curvatures of Polylines

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Abstract

Accurate estimations of geometric properties of a smooth curve from its discrete approximation are important for many computer graphics and computer vision applications. To assess and improve the quality of such an approximation, we assume that the curve is known in general form. Then we can represent the curve by a Taylor series expansion and compare its geometric properties with the corresponding discrete approximations. In turn we can either prove convergence of these approximations towards the true properties as the edge lengths tend to zero, or we can get hints on how to eliminate the error. In this paper, we propose and study discrete schemes for estimating tangent and normal vectors as well as for estimating curvature and torsion of a smooth 3D curve approximated by a polyline. Thereby we make some interesting findings about connections between (smooth) classical curves and certain estimation schemes for polylines.

CR Categories: I.3.5 [Computer Graphics]: Computational Geometry and Object Modeling;

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1 Introduction

Reliable approximations of differential properties of a curve form the basis of many algorithms in computer graphics and computer vision. Curvature, for example, can be used to define the smoothness of a curve. Furthermore, the understanding of discrete normals and curvatures of curves is a precondition for the even more important – and more difficult – task of understanding discrete normals and curvatures of a surface. In this sense, our work also lies the foundation for reliable estimates of normals and curvatures on meshes.

The problem of estimating differential properties of discrete approximations has already been treated in the classical literature of differential geometry [Sauer 1970]. But in that context, the speed of convergence was not an issue, and often very simple approximations were used, yielding, for example, only linear approximations for the tangent vector. Today, one common way to obtain tangent and normal vectors at a vertex of a polyline is to compute it as a weighted average of the incident edges (or as a weighted average of the edge normals, respectively). Various weights have been proposed for that purpose. The perhaps most popular schemes are uniform weighting, weighting by edge lengths, and weighting by inverse edge lengths. It was shown in [Anoshkina et al. 2002] that the last of these methods yields the best results for planar curves. But this result holds not necessarily for space curves, in particular there exists no unique edge normal from which the (uniquely defined)

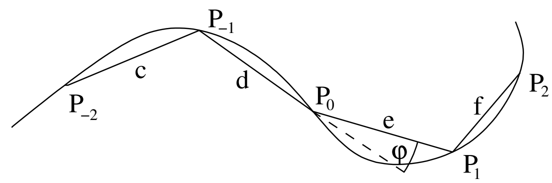

Figure 1: A space curve.

curve normal can be computed. For the estimation of curvature and torsion, various methods have been suggested by Boutin [2000]. With our approach, we yield simpler formulae which, nevertheless, exhibit at least the same accuracy.

There are basically two ways to evaluate the quality of any of these methods. On the one hand, they can be applied to a specific polyline interpolating an analytical curve, and the result can be compared to the exact tangent vector (or any other approximated geometric property) at the corresponding point. On the other hand, an asymptotic analysis can be applied. In this case, the analytical curve is given in general form, usually represented by a Taylor series expansion. Then the outcome of the discrete approximation can again be compared to the real tangent vector. Both methods have advantages and drawbacks. The first one cannot state general results, but only for certain test curves. The second method holds for all (analytical) curves and can give clues for design and improvement of the approximations. But it is only helpful for dense polylines where dense is not well-defined. It has successfully been applied for planar curves [Anoshkina et al. 2002]; for space curves, pioneering work has been done in [Boutin 2000].

In real world applications, all these computations have often to be done in the presence of noise. In this paper, we assume that all points lie exactly on a smooth curve since the definitions for differential properties are valid only in that case. Though we make this assumption for the development of our discrete approximation formulae, this does not mean that our work is useless for real data. The estimation error of every approximation scheme is composed of a systematic error inherent in the utilized approximation scheme and of an error introduced by noise. The goal of this paper is to minimize the former.

The main focus of this paper is developing a mathematical apparatus for the asymptotic analysis of arbitrary curves, and applying it to derive new, asymptotically correct estimations for tangents, normals, curvatures, and torsions of space curves. A uniform evaluation for existing approaches and our newly proposed approximations is given. In particular, we prove the convergence of our approximations and can show their optimality in many cases. To estimate torsion, which is a third derivative, we need at least four points for the approximation, but we consider also estimations using five points to obtain better results. The case of planar curves [Anoshkina et al. 2002] is consistently included.

2 Approximation of space curves

Let a smooth curve $\mathbf{r}$ be interpolated by the five points P_{-2}, P_{-1}, P_0, P_1, and P_2, with the corresponding edges $\overrightarrow{P_iP_{i+1}}$ denoted by $\mathbf{c}$, $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$, and their lengths denoted by c, d, e, and f, see Figure 1. Then these edges can be expressed by their Taylor expansions in the coordinate system given by the Frenet frame of $\mathbf{r}$ with tangent $\mathbf{t}$, normal $\mathbf{n}$, and binormal $\mathbf{b} = \mathbf{t} \times \mathbf{n}$. Let further κ denote the curvature at P_0 and τ denote the torsion at the same point. For the exact

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expansions have a look at Appendix A.

First, we use inverse edge lengths as weights for the edges:

2.1 Theorem (tangent vector). *The tangent of the circle passing through P_{-1}, P_0 and P_1 is a second order approximation of the real tangent of the curve* $\mathbf{r}$*:*

$$\begin{aligned}\tilde{\mathbf{t}} &:= \frac{de}{d+e}\Big(\frac{\mathbf{d}}{d^2}+\frac{\mathbf{e}}{e^2}\Big)\\ &= \mathbf{t}\Big(1-\frac{de}{8}\kappa^2+\frac{d^2e-de^2}{12}\kappa\kappa'+O(d,e)^4\Big)\\ &+\mathbf{n}\Big(\frac{de}{6}\kappa'-\frac{d^2e-de^2}{24}(\kappa''-\kappa\tau^2)+O(d,e)^4\Big)\\ &+\mathbf{b}\Big(-\frac{de}{6}\kappa\tau+\frac{d^2e-de^2}{24}(2\kappa'\tau+\kappa\tau')+O(d,e)^4\Big).\end{aligned} \tag{1}$$

This estimation is optimal among all three-point approximations of the tangent in the sense that the quadratic term in the normal component cannot be different from the one that shows up here. Also, this is the only linear combination of $\mathbf{d}$ *and* $\mathbf{e}$ *that yields a second order approximation.*

Proof. The equation can directly be derived from the Taylor expansions in Appendix A. If there were curves with other quadratic terms we could gain a tangent estimation and in turn an estimation of the normal for planar curves of the same accuracy, but this is not possible, see [Anoshkina et al. 2002].

The last statement of the proposition can easily be derived using the Taylor expansions of $\mathbf{d}$ and $\mathbf{e}$ from Appendix A. □

Note that in the planar case knowledge of tangent and normal is equivalent. Therefore, every tangent formula can be used to compute normals of plane curves. In 3D, the computation of normals is more difficult, however, because the oscillating plane is unknown. It can be done after estimating the binormals which determine that plane. A more direct approach is to compute the curvature vector, for example using finite differences.

2.2 Theorem (curvature vector). *The finite difference approach yields a linear approximation of the true curvature vector, and thus of the true normal vector. If all edges have equal length, the convergence is even quadratic.*

$$\begin{aligned}\bar{\mathbf{k}} &:= \frac{2}{d+e}\Big(\frac{\mathbf{e}}{e}-\frac{\mathbf{d}}{d}\Big)\\ &= \mathbf{t}\Big(\frac{d-e}{4}\kappa^2-\frac{d^2-de+e^2}{6}\kappa\kappa'+O(d,e)^3\Big),\\ &+\mathbf{n}\Big(\kappa-\frac{d-e}{3}\kappa'+\frac{d^2-de+e^2}{12}(\kappa''-\kappa\tau^2)+O(d,e)^3\Big)\\ &+\mathbf{b}\Big(\frac{d-e}{3}\kappa\tau-\frac{d^2-de+e^2}{12}(2\kappa'\tau+\kappa\tau')+O(d,e)^3\Big).\end{aligned} \tag{2}$$

Furthermore, this is the only linear combination of $\mathbf{d}$ *and* $\mathbf{e}$*, that yields a (at least) linear approximation of the real normal vector.*

Proof. Again all claims can directly be proven from the Taylor expansions given in Appendix A. □

From the curvature vector, we gain the curvature as the norm. Another possibility is to estimate the curvature by *angle approximation*. That approach is based on the definition of curvature as the rate of angular change of the tangent vector along the curve.

2.3 Theorem (curvature). *Let* φ *be the angle between* $\mathbf{d}$ *and* $\mathbf{e}$*, see Figure 1. Curvature, estimated using the discrete curvature vector (2) or angle approximation, respectively, converges linearly towards the true curvature. If all edges have equal length, the convergence is even quadratic.*

$$\begin{aligned}\bar{\kappa} &:= \|\bar{\mathbf{k}}\| = \kappa+\frac{e-d}{3}\kappa'+\frac{d^2-de+e^2}{12}\kappa''\\ &\quad-\frac{d^2+de+e^2}{36}\kappa\tau^2+\frac{d^2-2de+e^2}{32}\kappa^3+O(d,e)^3,\end{aligned}$$

$$\begin{aligned}\hat{\kappa} &:= \frac{2\varphi}{d+e} = \kappa+\frac{e-d}{3}\kappa'+\frac{d^2-de+e^2}{12}(\kappa''+\frac{\kappa^3}{2})\\ &\quad-\frac{d^2+de+e^2}{36}\kappa\tau^2+O(d,e)^3.\end{aligned}$$

These estimations are optimal among all three-point approximations in the sense that the linear terms cannot be different from the ones that show up here.

Proof. Again the equations can be derived from Appendix A and optimality can be reduced to the planar case [Anoshkina et al. 2002]. □

Yet another way to estimate the curvature is as the inverse of the radius of the circle passing through P_{-1}, P_0 and P_1. This has been done in [Boutin 2000] and yields

$$\widetilde{\kappa} = \kappa+\frac{e-d}{3}\kappa'+\frac{d^2-de+e^2}{12}\kappa''-\frac{d^2+de+e^2}{36}\kappa\tau^2+O(d,e)^3.$$

Since $\sin\varphi$ equals φ up to quadratic error, we can compute an approximation for κ as

$$\frac{2\|\mathbf{d}\times\mathbf{e}\|}{de(d+e)} \approx \hat{\kappa}$$

without significant loss of accuracy.

Also note that for $d=e$ the expansion of the angle approximation becomes

$$\hat{\kappa} = \kappa+\frac{e^2}{12}(\kappa''+\frac{\kappa^3}{2}-\kappa\tau^2)+O(e^4),$$

see Appendix A, and here the quadratic term vanishes for a special class of curves called elastica, characterized by minimizing the bending energy

$$\int \kappa^2 ds \longrightarrow \min$$

while fixing end points. They were first introduced by Euler [1744] and have applications in computer graphics as well as in computer vision today [Cerda et al. 2004; Mumford 1994; Horn 1983].

2.4 Theorem (Euler's elastica). *The curvature estimation* $\hat{\kappa}$ *converges of fourth order for elastica if all edges have equal length.*

In fact, the lower order error terms vanish for an even broader class of curves, see Appendix B for a derivation.

Binormals $\mathbf{b}_i$ at P_i can be estimated by the normal of the plane defined by three consecutive points, P_{i-1}, P_i, and P_{i+1}, for example $\mathbf{b}_0 = \frac{\mathbf{d}\times\mathbf{e}}{\|\mathbf{d}\times\mathbf{e}\|}$. Now we apply the method of angle approximation to these binormals to compute the torsion $\hat{\tau}_{\mathbf{e}}$ (located at the edge $\mathbf{e}$) from the angle $\eta_{\mathbf{e}}$ between $\mathbf{b}_0$ and $\mathbf{b}_1$, $\|\mathbf{b}_1\times\mathbf{b}_0\| = \sin\eta_{\mathbf{e}}$. But instead of taking the norm of the cross product, which is computationally rather expensive, and even worse, always yields positive values, whereas torsion is a signed property, we use the fact from the Frenet equations that $\frac{d\mathbf{b}}{ds} = \tau\mathbf{n}$. Therefore, $\mathbf{b}_1\times\mathbf{b}_0$ should approximately be orthogonal to $\mathbf{n}$ and is thus approximately aligned with $\mathbf{t}$, and we define $\hat{\eta}_{\mathbf{e}} := \langle\mathbf{b}_1\times\mathbf{b}_0,\tilde{\mathbf{t}}\rangle$ where $\tilde{\mathbf{t}}$ denotes the tangent approximation from equation (1). In fact $\hat{\eta}_{\mathbf{e}} = \eta_{\mathbf{e}}+O(d,e,f)^3$ (because $\eta_{\mathbf{e}}$ depends linearly on d, e and f, and $\sin\eta_{\mathbf{e}}$ approximates $\eta_{\mathbf{e}}$ up to second order). We define analogously $\hat{\eta}_{\mathbf{d}}$ from $\mathbf{b}_{-1}$ and $\mathbf{b}_0$ and get (see Appendix A for the Taylor expansion of $\hat{\eta}_{\mathbf{e}}$)

2.5 Theorem (torsion, four points). *Using four of the five points* P_{-2}*,* P_{-1}*,* P_0*,* P_1*, and* P_2*, torsion can be approximated linearly as follows:*

$$\begin{aligned}\hat{\tau}_{\mathbf{d}} &:= \frac{3\hat{\eta}_{\mathbf{d}}}{c+d+e} = \tau-\frac{c-e}{6}\frac{\kappa'}{\kappa}\tau-\frac{c+2d-e}{4}\tau'+O(c,d,e)^2,\\ \hat{\tau}_{\mathbf{e}} &:= \frac{3\hat{\eta}_{\mathbf{e}}}{d+e+f} = \tau+\frac{f-d}{6}\frac{\kappa'}{\kappa}\tau-\frac{d-2e-f}{4}\tau'+O(d,e,f)^2.\end{aligned}$$

It is interesting to compare the above estimation $\hat{\tau}_{\mathbf{e}}$ with the results from Boutin [2000], who uses the same four points. Let g be the distance $\|\overrightarrow{P_0P_2}\|$. Then

$$\widetilde{\tau}_1 = \tau+\frac{d-e+3g}{6}\frac{\kappa'}{\kappa}\tau+\frac{e-d+g}{4}\tau'+O(d,e,f)^2$$

$$\text{and}\quad \widetilde{\tau}_2 = \tau+\frac{d+e+g}{6}\frac{\kappa'}{\kappa}\tau+\frac{e-d+g}{4}\tau'+O(d,e,f)^2.$$

Our approximation is more symmetric in the sense that the first linear error term vanishes if all edge lengths are equal. By estimating torsion using the angle between $\mathbf{b}_{-1}$ and $\mathbf{b}_1$, we can even get an expression completely without linear terms if $d = e$ and $c = f$.

A better way to obtain such a symmetric expression is to take the (unique) weighted average of $\hat{\tau}_{\mathbf{d}}$ and $\hat{\tau}_{\mathbf{e}}$ such that the term involving τ' vanishes completely and the term involving $\frac{\kappa'}{\kappa}\tau$ vanishes for $d = e$ and $c = f$:

$$\begin{aligned}\hat{\tau} &:= \frac{1}{c+d+e+f}\big((f+2e-d)\tau_{\mathbf{d}} + (c+2d-e)\tau_{\mathbf{e}}\big)\\ &= \tau - \frac{ce-e^2+d^2-df}{3(c+d+e+f)}\,\frac{\kappa'}{\kappa}\tau + O(c,d,e,f)^2.\end{aligned}$$

It can be further improved by estimating $\frac{\kappa'}{\kappa}\tau$ and eliminating the corresponding error term. In that way, we can get a five-point approximation of the torsion at P_0 that converges quadratically for arbitrary edge lengths. For this purpose, we approximate curvatures at P_{-1} and P_1 from the angles φ_{-1} between $\mathbf{c}$ and $\mathbf{d}$, and φ_1 between $\mathbf{e}$ and $\mathbf{f}$:

$$\begin{aligned}\kappa_{-1} &:= \frac{2\varphi_{-1}}{c+d} = \kappa - \frac{c+2d}{3}\kappa' + O(c,d)^2\\ \text{and}\quad \kappa_1 &:= \frac{2\varphi_1}{e+f} = \kappa + \frac{f+2e}{3}\kappa' + O(e,f)^2.\end{aligned}$$

From this, we get five-point estimates for curvature

$$\begin{aligned}\kappa_5 &:= \frac{1}{c+2(d+e)+f}\left((2e+f)\kappa_{-1} + (c+2d)\kappa_1\right)\\ &= \kappa + O(c,d,e,f)^2,\end{aligned}$$

for its derivative (also suggested in [Boutin 2000])

$$\kappa_5' := \frac{3}{c+2(d+e)+f}\left(\kappa_1 - \kappa_{-1}\right) = \kappa' + O(c,d,e,f),$$

and finally

2.6 Theorem (torsion, five points). *Five points are sufficient to obtain a second order approximation for torsion:*

$$\tau_5 := \hat{\tau} + \frac{ce-df+d^2-e^2}{3(c+d+e+f)}\,\frac{\kappa_5'}{\kappa_5}\hat{\tau} = \tau + O(c,d,e,f)^2.$$

Here, $\hat{\tau}$, κ_5, and κ_5' are defined as above.

3 Conclusion

We have presented a mathematical framework to evaluate and develop approximation schemes to estimate differential properties of discrete curves. Its application yielded several formulae to estimate curvature, torsion, and the Frenet frame of a space curve, such that they converge towards their smooth counterparts as edge lengths tend to zero. Furthermore, we proved the optimality of our estimates in many cases. Thus, we provided a useful toolbox for the analysis of polylines in three-dimensional space.

In the future, we plan to extend our research to the asymptotic properties of estimations of normals and curvatures of meshes, and curvature and torsion of curves on surfaces. Also, we want to examine the influence of noise on normal and curvature estimations.

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A Taylor series expansion of space curves

In this appendix, we conduct an asymptotic analysis of an arbitrary curve $\mathbf{r}(s)$, with $\mathbf{r}(s_i) = P_i$, $s_0 = 0$, interpolated by a polyline as in Figure 1. Our treatment is based on the work of Anoshkina et al. [2002], but we have to take into account the higher complexity of three-dimensional space; in particular, the notion of torsion has no meaning for planar curves.

We assume without loss of generality that the curve is parameterized by arc length. This facilitates the problem to express discrete properties in geometrical meaningful terms like curvature and torsion by using Taylor series along with the well known Frenet equations [do Carmo 1976; Koenderink 1990]

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n}, \quad \frac{d\mathbf{b}}{ds} = \tau\mathbf{n}, \quad \frac{d\mathbf{n}}{ds} = -\kappa\mathbf{t} - \tau\mathbf{b}, \tag{3}$$

where $\mathbf{t}$, $\mathbf{n}$ and $\mathbf{b}$ are the unit tangent, the unit normal and the unit binormal vector, respectively, and κ and τ are curvature and torsion, respectively. (We omit the position s since the equations hold for all (fixed) s and we are interested only in the case $s = 0$, anyway.) Differentiating the curve $\mathbf{r}(s)$ then yields

$$\begin{aligned}\mathbf{r}' = \mathbf{t}, \quad \mathbf{r}'' = \mathbf{t}' = \kappa\mathbf{n}, \quad \mathbf{r}''' &= (\kappa\mathbf{n})' = \kappa'\mathbf{n} - \kappa^2\mathbf{t} - \kappa\tau\mathbf{b},\\ \mathbf{r}^{(4)} &= -3\kappa\kappa'\mathbf{t} + (\kappa'' - \kappa^3 - \kappa\tau^2)\mathbf{n} - (2\kappa'\tau + \kappa\tau')\mathbf{b},\\ \mathbf{r}^{(5)} = &\quad (\kappa^4 + \kappa^2\tau^2 - 4\kappa\kappa'' - 3(\kappa')^2)\mathbf{t}\\ &- (6\kappa^2\kappa' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \kappa''')\mathbf{n}\\ &+ (\kappa^3\tau + \kappa\tau^3 - \kappa\tau'' - 3\kappa'\tau' - 3\kappa''\tau)\mathbf{b},\end{aligned}$$

and so on. Now we can use Taylor expansion to express the edge $\mathbf{e} = \overrightarrow{P_0P_1} = \mathbf{r}(s_1) - \mathbf{r}(0)$ in the local canonical form [Kreyszig 1959; Sauer 1970]:

$$\begin{aligned}\mathbf{e} &= s_1\mathbf{r}' + \frac{s_1^2}{2}\mathbf{r}'' + \frac{s_1^3}{6}\mathbf{r}''' + \frac{s_1^4}{24}\mathbf{r}^{(4)} + \frac{s_1^5}{120}\mathbf{r}^{(5)} + O(s_1^6)\\ &= \mathbf{t}\Big(s_1 - \frac{s_1^3}{6}\kappa^2 - \frac{s_1^4}{8}\kappa\kappa'\\ &\qquad + \frac{s_1^5}{120}\big(\kappa^4 + \kappa^2\tau^2 - 4\kappa\kappa'' - 3(\kappa')^2\big) + O(s_1^6)\Big)\\ &\quad + \mathbf{n}\Big(\frac{s_1^2}{2}\kappa + \frac{s_1^3}{6}\kappa' + \frac{s_1^4}{24}(\kappa'' - \kappa^3 - \kappa\tau^2)\\ &\qquad - \frac{s_1^5}{120}(6\kappa^2\kappa' + 3\kappa\tau\tau' + 3\kappa'\tau^2 - \kappa''') + O(s_1^6)\Big)\\ &\quad + \mathbf{b}\Big(-\frac{s_1^3}{6}\kappa\tau - \frac{s_1^4}{24}(2\kappa'\tau + \kappa\tau')\\ &\qquad + \frac{s_1^5}{120}(\kappa^3\tau + \kappa\tau^3 - \kappa\tau'' - 3\kappa'\tau' - 3\kappa''\tau) + O(s_1^6)\Big).\end{aligned}$$

In the next step, we express $\mathbf{e}$ only in terms of its length e without using the possibly unknown geodesic length s_1. Since $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ is an orthonormal basis, we can compute e in terms of s_1 by

$$\begin{aligned}\|\mathbf{e}\|^2 &= s_1^2 - \frac{s_1^4}{12}\kappa^2 - \frac{s_1^5}{12}\kappa\kappa' \\ &\quad + \frac{s_1^6}{360}(\kappa^4 + \kappa^2\tau^2 - 9\kappa\kappa'' - 8(\kappa')^2) + O(s_1^7),\end{aligned}$$

$$\begin{aligned}e := \|\mathbf{e}\| &= s_1 - \frac{s_1^3}{24}\kappa^2 - \frac{s_1^4}{24}\kappa\kappa' \\ &\quad + \frac{s_1^5}{5760}(3\kappa^4 + 8\kappa^2\tau^2 - 72\kappa\kappa'' - 64(\kappa')^2) + O(s_1^6).\end{aligned}$$

After inverting the Taylor series for e, we obtain

$$\begin{aligned}s_1 &= e + \frac{e^3}{24}\kappa^2 + \frac{e^4}{24}\kappa\kappa' \\ &\quad + \frac{e^5}{5760}(27\kappa^4 - 8\kappa^2\tau^2 + 72\kappa\kappa'' + 64(\kappa')^2) + O(e^6).\end{aligned}$$

Substituting the expansion of s_1 into the formula for $\mathbf{e}$ and dividing by e yields

$$\begin{aligned}\frac{\mathbf{e}}{e} &= \mathbf{t}\Big(1 - \frac{e^2}{8}\kappa^2 - \frac{e^3}{12}\kappa\kappa' \\ &\qquad - \frac{e^4}{1152}(9\kappa^4 - 8\kappa^2\tau^2 + 24\kappa\kappa'' + 16(\kappa')^2) + O(e^5)\Big) \\ &\quad + \mathbf{n}\Big(\frac{e}{2}\kappa + \frac{e^2}{6}\kappa' + \frac{e^3}{24}(\kappa'' - \kappa\tau^2) \\ &\qquad + \frac{e^4}{240}(3\kappa^2\kappa' - 6\kappa\tau\tau' - 6\kappa'\tau^2 + 2\kappa''') + O(e^5)\Big) \\ &\quad + \mathbf{b}\Big(-\frac{e^2}{6}\kappa\tau - \frac{e^3}{24}(2\kappa'\tau + \kappa\tau') \\ &\qquad - \frac{e^4}{240}(3\kappa^3\tau - 2\kappa\tau^3 + 2\kappa\tau'' + 6\kappa'\tau' + 6\kappa''\tau) + O(e^5)\Big).\end{aligned}$$

In a similar fashion, we obtain $\frac{\mathbf{f}}{f}$ from the difference of two Taylor expansions, $\mathbf{f} = \mathbf{r}(s_1 + (s_2 - s_1)) - \mathbf{r}(s_1)$, and in the same way we get the expressions for $\frac{\mathbf{d}}{d}$ and $\frac{\mathbf{c}}{c}$.

Using these series, we can compute the cross product of $\frac{\mathbf{d}}{d}$ and $\frac{\mathbf{e}}{e}$:

$$\begin{aligned}\frac{\mathbf{d}}{d} \times \frac{\mathbf{e}}{e} &= \mathbf{t}\Big(\frac{d^2e + de^2}{12}\kappa^2\tau + O(d,e)^4\Big) \\ &\quad + \mathbf{n}\Big(\frac{e^2 - d^2}{6}\kappa\tau + \frac{d^3 + e^3}{24}(2\kappa'\tau + \kappa\tau') + O(d,e)^4\Big) \\ &\quad + \mathbf{b}\Big(\frac{d+e}{2}\kappa + \frac{e^2 - d^2}{6}\kappa' \\ &\qquad + \frac{d^3 + e^3}{24}(\kappa'' - \kappa\tau^2) - \frac{d^2e + de^2}{16}\kappa^3 + O(d,e)^4\Big).\end{aligned}$$

Note that the quadratic terms vanish for $d = e$. The same is true for fourth order terms:

$$\begin{aligned}\frac{\mathbf{d}}{d} \times \frac{\mathbf{e}}{e} &\overset{d=e}{=} \mathbf{t}\Big(\frac{e^3}{6}\kappa^2\tau + O(e^5)\Big) \\ &\quad + \mathbf{n}\Big(\frac{e^3}{12}(2\kappa'\tau + \kappa\tau') + O(e^5)\Big) \\ &\quad + \mathbf{b}\Big(e\kappa + \frac{e^3}{12}(\kappa'' - \kappa\tau^2) - \frac{e^3}{8}\kappa^3 + O(e^5)\Big).\end{aligned}$$

Since the norm of the above vector equals $\sin\varphi$, we obtain

$$\begin{aligned}\sin\varphi &= \frac{d+e}{2}\kappa - \frac{d^2 - e^2}{6}\kappa' + \frac{(d-e)(d^2-e^2)}{36}\kappa\tau^2 \\ &\quad + \frac{d^3 + e^3}{24}(\kappa'' - \kappa\tau^2) - \frac{d^2e + de^2}{16}\kappa^3 + O(d,e)^4,\end{aligned}$$

$$\begin{aligned}\varphi &= \frac{d+e}{2}\kappa - \frac{d^2 - e^2}{6}\kappa' + \frac{d^3 + e^3}{48}(\kappa^3 - \frac{2}{3}\kappa\tau^2 + 2\kappa'') \\ &\quad - \frac{d^2e + de^2}{36}\kappa\tau^2 + O(d,e)^4,\end{aligned}$$

and for $d = e$

$$\varphi \overset{d=e}{=} e\kappa + \frac{e^3}{24}(2\kappa'' + \kappa^3 - 2\kappa\tau^2) + O(e^5).$$

We can also compute the normalized binormal at P_0 by

$$\begin{aligned}\mathbf{b}_0 := \frac{\mathbf{d} \times \mathbf{e}}{\|\mathbf{d} \times \mathbf{e}\|} &= \frac{\frac{\mathbf{d}}{d} \times \frac{\mathbf{e}}{e}}{\sin\varphi} \\ &= \mathbf{t}\Big(\frac{de}{6}\kappa\tau + O(d,e)^3\Big) \\ &\quad + \mathbf{n}\Big(\frac{e-d}{3}\tau + \frac{d^2 - de + e^2}{12}\tau' + \frac{d^2 + de + e^2}{18}\frac{\kappa'}{\kappa}\tau + O(d,e)^3\Big) \\ &\quad + \mathbf{b}\Big(1 - \frac{(d-e)^2}{18}\tau^2 + O(d,e)^3\Big).\end{aligned}$$

The terms for the binormals $\mathbf{b}_{-1}$ at P_{-1} and $\mathbf{b}_1$ at P_1 are similar. With those in turn, we can estimate the angle between two consecutive binormals as

$$\begin{aligned}\hat{\eta}_{\mathbf{e}} = \langle \mathbf{b}_1 \times \mathbf{b}_0, \tilde{\mathbf{t}} \rangle &= \frac{d+e+f}{3}\tau - \frac{d^2 + de - ef - f^2}{18}\frac{\kappa'}{\kappa}\tau \\ &\quad - \frac{d^2 - de - 2e^2 - 3ef - f^2}{12}\tau' + O(d,e,f)^3\end{aligned}$$

where $\tilde{\mathbf{t}}$ is the tangent approximation from equation (1).

B Euler's elastica for space curves

In this section, we will derive necessary conditions for a space curve $\mathbf{r}(s)$ to be an elastica, that means

$$\int \kappa^2 ds \longrightarrow \min,$$

while fixing position and tangent of the two end points. Hereby, we follow the treatment given in [Belyaev et al. 1999] and [Mumford 1994] for elastica in the plane. Nevertheless, the situation for elastica in the three-dimensional space is more complex.

We consider a small perturbation of $\mathbf{r}(s)$

$$\hat{\mathbf{r}}(s) := \mathbf{r}(s) + \varepsilon\big(h(s)\mathbf{n} + k(s)\mathbf{b}\big)$$

where $\mathbf{r}(s)$ is an elastica parameterized by arc length s, $h(s)$ and $k(s)$ are real functions with compact support, and ε is a real number. Using the Frenet equations we get

$$\frac{d\hat{\mathbf{r}}}{ds} = \mathbf{t} + \varepsilon\big(-h\kappa\mathbf{t} + (h' + k\tau)\mathbf{n} + (k' - h\tau)\mathbf{b}\big).$$

Let $\hat{\mathbf{r}}(\hat{s})$ be a parameterization of $\hat{\mathbf{r}}$ by arc length. Then

$$d\hat{s} = \Big\|\frac{d\hat{\mathbf{r}}}{ds}\Big\| ds = (1 - \varepsilon h\kappa + O(\varepsilon^2))ds.$$

Therefore, we have

$$\hat{\mathbf{t}} = \frac{d\hat{\mathbf{r}}}{d\hat{s}} = \frac{d\hat{\mathbf{r}}}{ds}\frac{ds}{d\hat{s}} = \mathbf{t} + \varepsilon\big((h' + k\tau)\mathbf{n} + (k' - h\tau)\mathbf{b}\big) + O(\varepsilon^2),$$

$$\begin{aligned}\hat{\kappa}\hat{\mathbf{n}} &= \frac{d\hat{\mathbf{t}}}{d\hat{s}} = \frac{d\hat{\mathbf{t}}}{ds}\frac{ds}{d\hat{s}} \\ &= \kappa\mathbf{n} + \varepsilon(-h'\kappa - k\kappa\tau)\mathbf{t} + \varepsilon(h(\kappa^2 - \tau^2) + h'' + k\tau' + 2k'\tau)\mathbf{n} \\ &\quad + \varepsilon(-h\tau' - 2h'\tau - k\tau^2 + k'')\mathbf{b} + O(\varepsilon^2),\end{aligned}$$

and

$$\hat{\kappa}^2 = \|\hat{\kappa}\hat{\mathbf{n}}\|^2 = \kappa^2 + 2\varepsilon\kappa\big(h(\kappa^2 - \tau^2) + h'' + k\tau' + 2k'\tau\big) + O(\varepsilon^2).$$

Now we can compute, using integration by parts:

$$\begin{aligned}\int \hat{\kappa}^2 d\hat{s} &= \int \kappa^2 d\hat{s} \\ &\quad + \varepsilon\int h(2\kappa^3 + 2\kappa'' - 2\kappa\tau^2) - 2k(\kappa\tau' + 2\kappa'\tau)d\hat{s} + O(\varepsilon^2) \\ &= \int \kappa^2 ds + \varepsilon\int h(\kappa^3 + 2\kappa'' - 2\kappa\tau^2) - 2k(\kappa\tau' + 2\kappa'\tau)ds + O(\varepsilon^2).\end{aligned}$$

Because $h(s)$ and $k(s)$ are arbitrary functions with compact support and the integral $\int \kappa^2 ds$ is minimal for $\mathbf{r}(s)$, this shows:

$$\kappa'' + \frac{\kappa^3}{2} - \kappa\tau^2 = 0 \quad \text{and} \quad \kappa'\tau + \frac{\kappa\tau'}{2} = 0.$$