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THE GEOMETRY AND TOPOLOGY ON GRASSMANN MANIFOLDS

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Abstract. This paper shows that the Grassmann Manifolds $G_{\mathbf{F}}(n,N)$ can all be imbedded in an Euclidean space $M_{\mathbf{F}}(N)$ naturally and the imbedding can be realized by the eigenfunctions of Laplacian $\triangle$ on $G_{\mathbf{F}}(n,N)$. They are all minimal submanifolds in some spheres of $M_{\mathbf{F}}(N)$ respectively. Using these imbeddings, we construct some degenerate Morse functions on Grassmann Manifolds, show that the homology of the complex and quaternion Grassmann Manifolds can be computed easily.

1. Introduction

Let $G_{\mathbf{F}}(n,N)$ be the Grassmann manifold formed by all n-subspaces in $\mathbf{F}^N$, where $\mathbf{F}$ is the set of real numbers, complex numbers or quaternions. The manifold $G_{\mathbf{F}}(n,N)$ is a symmetric space (see [7] or [8]). The Grassmann manifolds are important in the study of the geometry and the topology, especially in the theory of fibre bundles.

Let $\widetilde{G}(n,N)$ be the oriented Grassmann manifold formed by all oriented n dimensional subspaces of $\mathbf{R}^N$. In [3], Chen showed that $\widetilde{G}(n,N)$ can be imbedded in the unit sphere of wedge product space $\bigwedge^n(\mathbf{R}^N)$ as a minimal submanifold. Takahashi [10] proved that a compact homogenous Riemannian manifold with irreducible linear isotropy group admits a minimal immersion into an Euclidean sphere, see also Takeuchi and Kobayashi [11].

Let $M_{\mathbf{F}}(N)$ be the set of $N\times N$ matrices A with values in $\mathbf{F}$ such that $\overline{A}^t = A$. $M_{\mathbf{F}}(N)$ is an Euclidean space. Let $M_{\mathbf{F}}(n,N) = \{A \in M_{\mathbf{F}}(N) \mid A^2 = A,\ \mathrm{r}(A) = n\}$ be a subspace of $M_{\mathbf{F}}(N)$, where $\mathrm{r}(A)$ be the rank of the matrix A. The matrix $A \in M_{\mathbf{F}}(n,N)$ can be viewed as a projection on Euclidean space $\mathbf{F}^N$.

For any $\pi \in G_{\mathbf{F}}(n,N)$, let $e_1,\cdots,e_n$ be an orthonormal basis of π. Then $(e_1,\cdots,e_n)$ is an $N\times n$ matrix. Define

$$\varphi(\pi) = (e_1,\cdots,e_n)\overline{(e_1,\cdots,e_n)}^t = \sum_i e_i\bar{e}_i^t.$$

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We show in §2, the map $\varphi:\ G_{\mathbf{F}}(n,N)\to M_{\mathbf{F}}(N)$ is an imbedding and we have $\varphi(G_{\mathbf{F}}(n,N))=M_{\mathbf{F}}(n,N)$. Then

$$\bigcup_{n=0}^{N}\varphi(G_{\mathbf{F}}(n,N))=\{A\in M_{\mathbf{F}}(N)\mid A^2=A\}.$$

Let $\{A\in M_{\mathbf{F}}(N)\mid \mathrm{tr}A=n\}$ be a hyperplane in $M_{\mathbf{F}}(N)$ and $S(\sqrt{n})$ the sphere of $M_{\mathbf{F}}(N)$ with radius $\sqrt{n}$. In §2, we also show that $M_{\mathbf{F}}(n,N)$ is a minimal submanifold in the sphere $S(\sqrt{n})\bigcap\{A\in M_{\mathbf{F}}(N)\mid \mathrm{tr}A=n\}$. These minimal submanifolds are the natural generalization of the famous Veronese surface.

Let $G_{\mathbf{F}}(N)$ be the group which preserving the inner product on Euclidean space $\mathbf{F}^N$. With the spaces $M_{\mathbf{F}}(n,N)$, we can show that the Grassmann manifold $G_{\mathbf{F}}(n,N)$ can be imbedded in the group $G_{\mathbf{F}}(N)$.

In §3, we construct some degenerate Morse functions on Grassmann Manifolds. Show that the Poincaré polynomial of $G_{\mathbf{F}}(n,N)$ can be represented by

$$P_t(G_{\mathbf{F}}(n,N))=P_t(G_{\mathbf{F}}(n,N-1))+t^{c(N-n)}P_t(G_{\mathbf{F}}(n-1,N-1)),$$

where $\mathbf{F}=\mathbf{C}$ or $\mathbf{F}=\mathbf{H}$ and c the dimension of $\mathbf{F}$. Then the homology of the complex and quaternion Grassmann Manifolds can be computed easily in low dimensional cases.

These results are consistent with the results computed by using Schubert variety. As in [4] or [5], we consider the case of $\mathbf{F}=\mathbf{C}$. Let

$$0\le a_0\le a_1\le\cdots\le a_n\le N-n$$

be a sequence of integers. There is a natural one-one correspondence between the set of $(a_0,a_1,\cdots,a_n)$ and the generators of the homology $H_*(G_{\mathbf{C}}(n,N))$. The dimension of $(a_0,a_1,\cdots,a_n)$ is $2(a_0+a_1+\cdots+a_n)$. Such elements can be divided into two classes:

(1) $(a_0,a_1,\cdots,a_n)$, where $a_n\le N-n-1$;

(2) $(a_0,a_1,\cdots,a_{n-1},N-n)$, where $a_{n-1}\le N-n$.

These also show

$$P_t(G_{\mathbf{C}}(n,N))=P_t(G_{\mathbf{C}}(n,N-1))+t^{2(N-n)}P_t(G_{\mathbf{C}}(n-1,N-1)).$$

The Poincaré polynomial of $G_{\mathbf{F}}(n,N)$ can also be represented by

$$\begin{aligned}P_t(G_{\mathbf{F}}(n,N))=\ & t^{cn}P_t(G_{\mathbf{F}}(n,N-2))+t^{c(N-n)}P_t(G_{\mathbf{F}}(n-2,N-2))\\ &+(1+t^{c(N-1)})P_t(G_{\mathbf{F}}(n-1,N-2)),\end{aligned}$$

where $n,N-n\ge 2$, $\mathbf{F}=\mathbf{C}$ or $\mathbf{H}$.

2. The minimal imbedding of $G_{\mathbf{F}}(n,N)$ in the sphere

Let $\mathbf{F}$ be the set of real numbers $\mathbf{R}$, complex numbers $\mathbf{C}$ or quaternions $\mathbf{H}$. The quaternions $\mathbf{H}$ is generated by $i, j, k = ij$. For any $u \in \mathbf{F}$, $\overline{u}$ is the conjugation of u ($\overline{u \cdot v} = \overline{v} \cdot \overline{u}$ if $u, v \in \mathbf{H}$). For any $\lambda \in \mathbf{F}$, λ acts on the right of $u = (u_1, \cdots, u_N)^t \in \mathbf{F}^N$. For any $u = (u_1, \cdots, u_N)^t$, $v = (v_1, \cdots, v_N)^t \in \mathbf{F}^N$,

$$(u, v) = \bar{v}^t \cdot u = \sum_A \bar{v}_A u_A$$

defines an inner product on $\mathbf{F}^N$. Let $G_{\mathbf{F}}(N)$ be the group acting on the left of $\mathbf{F}^N$ which preserving the inner product $(,)$ on $\mathbf{F}^N$. If $\mathbf{F} = \mathbf{R}$, $G_{\mathbf{F}}(N) = O(N)$ is the orthogonal group; if $\mathbf{F} = \mathbf{C}$, $G_{\mathbf{F}}(N) = U(N)$ is the complex unitary group; if $\mathbf{F} = \mathbf{H}$, $G_{\mathbf{F}}(N) = Sp(N)$ is the symplectic group.

Let $G_{\mathbf{F}}(n, N) \approx \frac{G_{\mathbf{F}}(N)}{G_{\mathbf{F}}(n) \times G_{\mathbf{F}}(N-n)}$ be the Grassmann manifold formed by all subspaces in $\mathbf{F}^N$ of dimension n. Let $e_1, \cdots, e_n, e_{n+1}, \cdots, e_N$ be orthonormal frame fields on $G_{\mathbf{F}}(n, N)$ such that the element of $G_{\mathbf{F}}(n, N)$ is generated by $e_1, \cdots, e_n$ locally. By the method of moving frame, there are local 1-forms ω_A^B defined by

$$de_A = \sum_B e_B \omega_A^B, \ \omega_A^B + \overline{\omega}_B^A = 0, \quad A, B = 1, \cdots, N.$$

Restricting the two form $\Phi = \text{Re } (\sum\limits_{i,\alpha} \omega_i^\alpha \overline{\omega}_i^\alpha)$ on $G_{\mathbf{F}}(n, N)$ defines a Riemannian metric (see [4]). Unless otherwise stated, we agree on the following arranges of the indices:

$$1 \le i, j, \cdots \le n, \quad n+1 \le \alpha, \beta, \cdots \le N, \quad 1 \le A, B, \cdots \le N.$$

Let $M_{\mathbf{F}}(N)$ be the set of $N \times N$ matrices A with values in $\mathbf{F}$ such that $\overline{A}^t = A$. With the inner product defined by

$$\langle A, B \rangle = \text{Re tr}(AB) = \sum_A x_{AA} y_{AA} + 2\text{Re} \sum_{A<B} x_{AB} \bar{y}_{AB},$$

$A = (x_{AB}), B = (y_{AB}) \in M_{\mathbf{F}}(N)$, $M_{\mathbf{F}}(N)$ becomes an Euclidean space. The real dimension of $M_{\mathbf{F}}(N)$ is $N + \frac{1}{2}cN(N-1)$, where c is the real dimension of $\mathbf{F}$.

Lemma 2.1. *Let $e_1, \cdots, e_N$ be an orthonormal frame on $\mathbf{F}^N$. The following elements form an orthogonal basis of $M_{\mathbf{F}}(N)$ with respect to the norm $\langle , \rangle$ respectively,*

(1) $e_A e_A^t$, $e_B e_C^t + e_C e_B^t$, *when* $\mathbf{F} = \mathbf{R}$;

(2) $e_A \bar{e}_A^t$, $e_B \bar{e}_C^t + e_C \bar{e}_B^t$, $e_B i \bar{e}_C^t - e_C i \bar{e}_B^t$, *when* $\mathbf{F} = \mathbf{C}$;

(3) $e_A \bar{e}_A^t$, $e_B \bar{e}_C^t + e_C \bar{e}_B^t$, $e_B i \bar{e}_C^t - e_C i \bar{e}_B^t$, $e_B j \bar{e}_C^t - e_C j \bar{e}_B^t$, $e_B k \bar{e}_C^t - e_C k \bar{e}_B^t$, *when* $\mathbf{F} = \mathbf{H}$,

where $A, B, C = 1, \cdots, N,\ B < C$.

Proof. The proof is a direct computation. For example, we have

$$\langle e_A \bar{e}_A^t, e_B \bar{e}_C^t + e_C \bar{e}_B^t \rangle = \text{Re tr}(e_A \bar{e}_C^t \delta_{AB} + e_A \bar{e}_B^t \delta_{AC}) = 2\delta_{AB}\delta_{AC},$$

and

$$\langle e_B \bar{e}_C^t + e_C \bar{e}_B^t, e_B i \bar{e}_C^t - e_C i \bar{e}_B^t \rangle = \text{Re tr}(e_C i \bar{e}_C^t - e_B i \bar{e}_B^t) = 0.$$

□

Note that the basis of $M_{\mathbf{F}}(N)$ described in Lemma 2.1 are all real.

For any $\pi \in G_{\mathbf{F}}(n, N)$, let $e_1, \cdots, e_n$ be an orthonormal basis of π. Then $(e_1, \cdots, e_n)$ is an $N \times n$ matrix. Define

$$\varphi: \ G_{\mathbf{F}}(n, N) \to M_{\mathbf{F}}(N),$$
$$\varphi(\pi) = (e_1, \cdots, e_n)\overline{(e_1, \cdots, e_n)}^t = \sum_i e_i \bar{e}_i^t.$$

It is easy to see that $\varphi(\pi)$ is independent of the choice of the orthonormal basis $e_1, \cdots, e_n$. Let $M_{\mathbf{F}}(n, N) = \{A \in M_{\mathbf{F}}(N) \mid A^2 = A,\ \mathrm{r}(A) = n\}$ be a subspace of $M_{\mathbf{F}}(N)$, where $\mathrm{r}(A)$ be the rank of matrix A.

Lemma 2.2. *The map* $\varphi: \ G_{\mathbf{F}}(n, N) \to M_{\mathbf{F}}(N)$ *is an imbedding and we have* $\varphi(G_{\mathbf{F}}(n, N)) = M_{\mathbf{F}}(n, N)$*. The induced metric on* $G_{\mathbf{F}}(n, N)$ *defined by* φ *is*

$$2\Phi = 2Re\ (\sum_{i,\alpha}\ \omega_i^\alpha \overline{\omega}_i^\alpha).$$

Proof. It is easy to see that $\varphi(G_{\mathbf{F}}(n, N)) \subset M_{\mathbf{F}}(n, N)$. On the other hand, the element $A \in M_{\mathbf{F}}(n, N)$ can be viewed as a projection on $\mathbf{F}^N$. Let $\pi = \{Ax \mid x \in \mathbf{F}^N\}$ be a subspace of $\mathbf{F}^N$ and $e_1, \cdots, e_n$ be an orthonormal basis of π. Therefore $Ae_i = e_i$. It is easy to see that $A = \sum_i e_i \bar{e}_i^t$ and $\varphi(\pi) = A$. Then we can identify $M_{\mathbf{F}}(n, N)$ with $G_{\mathbf{F}}(n, N)$. Let $e_1, \cdots, e_n, e_{n+1}, \cdots, e_N$ be orthonormal frame fields on $\mathbf{F}^N$ such that $G_{\mathbf{F}}(n, N)$ is generated by $e_1, \cdots, e_n$ locally. Hence

$$d\varphi = d\sum_i e_i \bar{e}_i^t = \sum_{i,\alpha} e_\alpha \omega_i^\alpha \bar{e}_i^t + \sum_{i,\alpha} e_i \bar{\omega}_i^\alpha \bar{e}_\alpha^t.$$

We compute the case of $\mathbf{F} = \mathbf{H}$ as an example, the other cases are similar.

Let $\omega_i^\alpha = a_i^\alpha + i b_i^\alpha + j c_i^\alpha + k d_i^\alpha$, where $a_i^\alpha, b_i^\alpha, c_i^\alpha, d_i^\alpha$ are real 1-forms. Then

$$\begin{aligned} d\varphi = \ & \sum_{i,\alpha} a_i^\alpha (e_\alpha \bar{e}_i^t + e_i \bar{e}_\alpha^t) + \sum_{i,\alpha} b_i^\alpha (e_\alpha i \bar{e}_i^t - e_i i \bar{e}_\alpha^t) \\ & + \sum_{i,\alpha} c_i^\alpha (e_\alpha j \bar{e}_i^t - e_i j \bar{e}_\alpha^t) + \sum_{i,\alpha} d_i^\alpha (e_\alpha k \bar{e}_i^t - e_i k \bar{e}_\alpha^t), \end{aligned}$$

and

$$e_\alpha \bar{e}_i^t + e_i \bar{e}_\alpha^t,\ e_\alpha i \bar{e}_i^t - e_i i \bar{e}_\alpha^t,\ e_\alpha j \bar{e}_i^t - e_i j \bar{e}_\alpha^t,\ e_\alpha k \bar{e}_i^t - e_i k \bar{e}_\alpha^t$$

form a basis of tangent space $TG_{\mathbf{H}}(n, N)$. By Lemma 2.1, these vectors are orthogonal with respect to the inner product $\langle , \rangle$. The norms of these vectors are all $\sqrt{2}$. Then

$$\langle d\varphi, d\varphi \rangle = 2 \sum_{i,\alpha} (a_i^\alpha \otimes a_i^\alpha + b_i^\alpha \otimes b_i^\alpha + c_i^\alpha \otimes c_i^\alpha + d_i^\alpha \otimes d_i^\alpha) = 2\Phi.$$

□

For any $A \in M_{\mathbf{F}}(n, N)$,

$$\langle A, A \rangle = \mathrm{tr}\ A\overline{A}^t = \mathrm{tr}A = \mathrm{tr}[(e_1, \cdots, e_n)\overline{(e_1, \cdots, e_n)}^t] = n,$$

then $|A| = \sqrt{n}$. These also show $\mathrm{r}(A) = \mathrm{tr}A$ for any $A \in M_{\mathbf{F}}(N)$ with $A^2 = A$. Therefore we also have

$$M_{\mathbf{F}}(n, N) = \{A \in M_{\mathbf{F}}(N) \mid A^2 = A,\ \mathrm{tr}A = n\},$$

and

$$\bigcup_{n=0}^{N} \varphi(G_{\mathbf{F}}(n, N)) = \{A \in M_{\mathbf{F}}(N) \mid A^2 = A\}.$$

Let I_N be the identity matrix of order N. For $A \in M_{\mathbf{F}}(n, N)$, we have $(I_N - A)^2 = I_N - A$, $\mathrm{r}(I_N - A) = \mathrm{tr}(I_N - A) = N - n$, hence $I_N - A \in M_{\mathbf{F}}(N - n, N)$. These show the map $A \to I_N - A$ gives an isometry between the manifolds $G_{\mathbf{F}}(n, N)$ and $G_{\mathbf{F}}(N - n, N)$.

As show above, for any $A \in M_{\mathbf{F}}(n, N)$, $|A|^2 = \mathrm{tr}A = n$, then $M_{\mathbf{F}}(n, N)$ is in the sphere $S(\sqrt{n}) = \{B \in M_{\mathbf{F}}(N) \mid |B|^2 = n\}$. By $\mathrm{tr}A = n$, we know that $M_{\mathbf{F}}(n, N)$ also in the hyperplane $\{B \in M_{\mathbf{F}}(N) \mid \mathrm{tr}B = n\}$. This hyperplane can also be defined by $\{B \in M_{\mathbf{F}}(N) \mid \langle B, I_N \rangle = n\}$. Then the normal vector of this hyperplane is $I_N = \sum\limits_i e_i \bar{e}_i^t + \sum\limits_\alpha e_\alpha \bar{e}_\alpha^t$.

When $\mathbf{F} = \mathbf{R}$, any element $A \in M_{\mathbf{R}}(1, 3) \approx \mathbf{R}P^2$ can be represented by

$$A = \begin{pmatrix} x_1^2 & x_1x_2 & x_1x_3 \\ x_2x_1 & x_2^2 & x_2x_3 \\ x_3x_1 & x_3x_2 & x_3^2 \end{pmatrix}, \quad x_1^2 + x_2^2 + x_3^2 = 1.$$

The map

$$B = (x_{AB}) \in M_{\mathbf{R}}(3) \to (x_{11}, x_{22}, x_{33}, \sqrt{2}x_{12}, \sqrt{2}x_{13}, \sqrt{2}x_{23}) \in \mathbf{R}^6$$

gives an isometry between these two Euclidean spaces. Then $M_{\mathbf{R}}(1, 3)$ is the famous Veronese surface.

Theorem 2.3. *The manifold $M_{\mathbf{F}}(n, N)$ is a minimal submanifold in the sphere $S(\sqrt{n}) \bigcap \{B \in M_{\mathbf{F}}(N) \mid \mathrm{tr}B = n\}$.*

Proof. With the notations used above,

$$d\varphi = \sum e_\alpha \omega_i^\alpha \bar{e}_i^t + \sum e_i \bar{\omega}_i^\alpha \bar{e}_\alpha^t,$$

$$d^2\varphi = \cdots + \sum [e_j \omega_\alpha^j \omega_i^\alpha \bar{e}_i^t + e_\alpha \omega_i^\alpha \bar{\omega}_i^\beta \bar{e}_\beta^t + e_\beta \omega_i^\beta \bar{\omega}_i^\alpha \bar{e}_\alpha^t + e_i \bar{\omega}_i^\alpha \bar{\omega}_\alpha^j \bar{e}_j^t],$$

where "$\cdots$" is the part of $d^2\varphi$ which tangent to $M_{\mathbf{F}}(n, N)$. Then the second fundamental form of the imbedding φ is

$$II = -\sum [e_j \bar{\omega}_j^\alpha \omega_i^\alpha \bar{e}_i^t + e_i \bar{\omega}_i^\alpha \omega_j^\alpha \bar{e}_j^t] + \sum [e_\alpha \omega_i^\alpha \bar{\omega}_i^\beta \bar{e}_\beta^t + e_\beta \omega_i^\beta \bar{\omega}_i^\alpha \bar{e}_\alpha^t].$$

The mean curvature vector is

$$H = -\frac{N-n}{n(N-n)} \sum_i e_i \bar{e}_i^t + \frac{n}{n(N-n)} \sum_\alpha e_\alpha \bar{e}_\alpha^t.$$

On the other hand, $\sum_i e_i \bar{e}_i^t$ and $I_N = \sum_i e_i \bar{e}_i^t + \sum_\alpha e_\alpha \bar{e}_\alpha^t$ are the normal vectors on $S(\sqrt{n})$ and $\{B \in M_{\mathbf{F}}(N) \mid \mathrm{tr} B = n\}$ at $\sum_i e_i \bar{e}_i^t$ respectively. These show $M_{\mathbf{F}}(n, N)$ is a minimal submanifold in the sphere $S(\sqrt{n}) \bigcap \{B \in M_{\mathbf{F}}(N) \mid \mathrm{tr} B = n\}$. □

The radius of the sphere $S(\sqrt{n}) \bigcap \{B \in M_{\mathbf{F}}(N) \mid \mathrm{tr} B = n\}$ is $\sqrt{\frac{n(N-n)}{N}}$.

The above proof also shows, the second fundamental form of $M_{\mathbf{F}}(n, N)$ has constant length [6]. The isometry group $G_{\mathbf{F}}(N)$ acts on the Grassmann manifold $G_{\mathbf{F}}(n, N)$ naturally. For any $g \in G_{\mathbf{F}}(N)$, $A \in M_{\mathbf{F}}(N)$, $Ad(g)A = gA\bar{g}^t$ defines an action of $G_{\mathbf{F}}(N)$ on $M_{\mathbf{F}}(N)$. Furthermore, the following diagram is commutative

$$\begin{array}{ccc} G_{\mathbf{F}}(n, N) & \xrightarrow{\varphi} & M_{\mathbf{F}}(N) \\ g \downarrow & & \downarrow Ad(g) \\ G_{\mathbf{F}}(n, N) & \xrightarrow{\varphi} & M_{\mathbf{F}}(N). \end{array}$$

Let $\triangle = (d+\delta)^2$ be the Laplacian on $G_{\mathbf{F}}(n, N)$ with respect to the metric 2Φ. For any $A \in M_{\mathbf{F}}(N)$, $f(\pi) = \langle \varphi(\pi), A \rangle$ is a function on the Grassmann manifold $G_{\mathbf{F}}(n, N)$. As is well-known, we have

$$\triangle f = -cn(N-n)\langle H, A \rangle.$$

As show above, $M_{\mathbf{F}}(n, N)$ is in the hyperplane $\{B \in M_{\mathbf{F}}(N) \mid \langle B, I_N \rangle = n\}$ of Euclidean space $M_{\mathbf{F}}(N)$. For any vector A parallel to this hyperplane, we have

$$\langle A, I_N \rangle = \langle A, \sum e_i \bar{e}_i^t \rangle + \langle A, \sum e_\alpha \bar{e}_\alpha^t \rangle = 0.$$

Then for such A, we have

$$\triangle f = cN \langle \sum_i e_i \bar{e}_i^t, A \rangle = cNf.$$

We have proved the following

Theorem 2.4. *The imbedding* $\varphi: G_{\mathbf{F}}(n,N) \to \{B \in M_{\mathbf{F}}(N) \mid trB = n\}$ *is formed by the eigenfunctions of Laplacian* $\triangle$ *on* $G_{\mathbf{F}}(n,N)$ *with eigenvalue* cN.

For any $A = \sum e_i \bar{e}_i^t \in M_{\mathbf{F}}(n,N)$, let $\widetilde{A} = I_N - 2A = I_N - 2\sum e_i \bar{e}_i^t$. It is easy to see that $\overline{\widetilde{A}}^t = \widetilde{A}$, $\widetilde{A}^2 = I_N$. Then $A \to \widetilde{A}$ gives a map $\psi: M_{\mathbf{F}}(n,N) \to G_{\mathbf{F}}(N)$.

It is interesting to note that when $\mathbf{F} = \mathbf{R}$ be the real numbers, the imbedding of $G_{\mathbf{R}}(n,N)$ in orthogonal group $O(N)$ defined above can be obtained by using Clifford algebra. Let $C\ell_N$ be the Clifford algebra associated to the Euclidean space $\mathbf{R}^N$ and $Pin(N)$ be the Pin group. Any unit vector v of $\mathbf{R}^N$ defines a reflection f_v on $\mathbf{R}^N$:

$$f_v(e) = v \cdot e \cdot v = e - 2(e,v)v, \quad \forall e \in \mathbf{R}^N,$$

where '$\cdot$' denotes the Clifford product. With the standard basis of $\mathbf{R}^N$, the map f_v can be represented by matrix $I_N - 2vv^t \in O(N)$.

Let $\widetilde{G}_{\mathbf{R}}(n,N)$ be the oriented Grassmann manifold. For any $\widetilde{\pi} \in \widetilde{G}_{\mathbf{R}}(n,N)$, we choose an oriented orthonormal basis $e_1, \cdots, e_n$ of $\widetilde{\pi}$. Note that $f_{e_i} f_{e_j} = f_{e_j} f_{e_i}$ for any i,j. Then $e_1 \cdot e_2 \cdots e_n \in Pin(N)$ and by the maps $\widetilde{G}_{\mathbf{R}}(n,N) \to Pin(N) \xrightarrow{Ad} O(N)$, we have a map

$$\widetilde{G}_{\mathbf{R}}(n,N) \to O(N), \quad \widetilde{\pi} \to f_{e_1} f_{e_2} \cdots f_{e_n},$$

$$\begin{aligned} f_{e_1} f_{e_2} \cdots f_{e_n} &= (I_N - 2e_1 e_1^t)(I_N - 2e_2 e_2^t) \cdots (I_N - 2e_n e_n^t) \\ &= I_N - 2\sum_{i=1}^{n} e_i e_i^t. \end{aligned}$$

The map $\widetilde{G}_{\mathbf{R}}(n,N) \to O(N)$ is an immersion. As $\det(I_N - 2\sum_{i=1}^{n} e_i e_i^t) = (-1)^n$ is constant, we can imbed real Grassmann manifold $G_{\mathbf{R}}(n,N)$ in $SO(N)$.

3. The Morse functions on the Grassmann manifolds

In [12], we have constructed many (degenerate or non-degenerate) Morse functions on the real oriented Grassmann manifolds by using calibrations. In the following we construct Morse functions on the Grassmann manifolds $G_{\mathbf{F}}(n,N)$.

For any $A \in M_{\mathbf{F}}(N)$, the map $f: G_{\mathbf{F}}(n,N) \to \mathbf{R}$, $f(\pi) = \langle \varphi(\pi), A \rangle$, defines a function on Grassmann manifolds $G_{\mathbf{F}}(n,N)$. f is a Morse function for almost every vector $A \in M_{\mathbf{F}}(N)$. But in general, it is difficult to find such element. First, we give some known results.

Let E_{AB} be the elements in $M_{\mathbf{F}}(N)$, $A \geq B$, where the entries in row A, column B and row B, column A are 1, the others are zero.

When $\mathbf{F} = \mathbf{C}$, $n = 1$, $G_{\mathbf{C}}(1, N) \approx \mathbf{C}P^{N-1}$ is the complex projective space. Let $A = \sum\limits_{A} c_A E_{AA} \in M_{\mathbf{C}}(N)$, $c_1 > c_2 > \cdots > c_N > 0$. For any $\pi \in G_{\mathbf{C}}(1, N)$, $\varphi(\pi) = e_1 \bar{e}_1^t$, $e_1 = (z_1, z_2, \cdots, z_N)^t$, $\sum\limits_{A} |z_A|^2 = 1$, then

$$f(\pi) = \langle \varphi(\pi), A \rangle = \sum_{A} c_A |z_A|^2.$$

As is well-known ([9]), f is a perfect Morse function on $\mathbf{C}P^{N-1}$.

Similar results hold for the real projective space $G_{\mathbf{R}}(1, N) \approx \mathbf{R}P^{N-1}$ and the quaternion projective space $G_{\mathbf{H}}(1, N) \approx \mathbf{H}P^{N-1}$. In the real case, the functions are not perfect.

The map $f: \ G_{\mathbf{F}}(n, N) \to \mathbf{R}$, $f(\pi) = \langle \varphi(\pi), E_{11} \rangle$, defines a function on $G_{\mathbf{F}}(n, N)$. To study this function we define two submanifolds of $G_{\mathbf{F}}(n, N)$. Let $G_{\mathbf{F}}(n-1, N-1)$ be a submanifold of $G_{\mathbf{F}}(n, N)$ such that every element of $G_{\mathbf{F}}(n-1, N-1)$ contains the vector $\tilde{e}_1 = (1, 0, \cdots, 0)^t \in \mathbf{F}^N$. Let $\mathbf{F}^{N-1} = \{u = (0, u_2, \cdots, u_N)^t \in \mathbf{F}^N\}$ be a subspace of $\mathbf{F}^N$ and $G_{\mathbf{F}}(n, N-1)$ be a submanifold of $G_{\mathbf{F}}(n, N)$ generated by the n-dimensional subspaces of $\mathbf{F}^{N-1}$.

Theorem 3.1. *The function $f : \ G_{\mathbf{F}}(n, N) \to \mathbf{R}$ is a degenerate Morse function on $G_{\mathbf{F}}(n, N)$, where $f(\pi) = \langle \varphi(\pi), E_{11} \rangle$. The critical submanifolds are $f^{-1}(0) = G_{\mathbf{F}}(n, N-1)$ and $f^{-1}(1) = G_{\mathbf{F}}(n-1, N-1)$ with indices* 0 *and $c(N-n)$ respectively.*

Proof. Let $e_1, \cdots, e_n, e_{n+1}, \cdots, e_N$ be orthonormal frame fields on $\mathbf{F}^N$ such that $G_{\mathbf{F}}(n, N)$ be generated by $e_1, \cdots, e_n$ locally. We have $f(\pi) = \sum\limits_{i=1}^{n} x_{i1} \bar{x}_{i1}$, where $e_i = (x_{i1}, \cdots, x_{iN})^t$. Then, $0 \leq f \leq 1$ and π is a critical point of function f if and only if

$$df = \sum_{i,\alpha} \langle e_\alpha \omega_i^\alpha \bar{e}_i^t + e_i \bar{\omega}_i^\alpha \bar{e}_\alpha^t, E_{11} \rangle = 0.$$

We prove the theorem for the real case, the other cases can be proved similarly, see the proof of Theorem 3.5. By Lemma 2.2, $df = 0$ if and only if

$$\langle e_\alpha e_i^t + e_i e_\alpha^t, E_{11} \rangle = 0$$

for any i, α. For $e_A = (x_{A1}, \cdots, x_{AN})$, $A = 1, \cdots, N$, we can assume $x_{i1} = 0$ for $i > 1$ and $x_{\alpha 1} = 0$ for $\alpha > n+1$. Obviously, we have $x_{11}^2 + x_{n+1\ 1}^2 = 1$.

Then

$$\langle e_\alpha e_i^t + e_i e_\alpha^t, E_{11}\rangle = 2x_{11}x_{n+1\ 1}\delta_{1i}\delta_{\alpha n+1},$$

and the point $\pi \in G_{\mathbf{R}}(n,N)$ is a critical point if and only if $x_{11} = 0$ or $x_{11} = 1$.

Let $\mathbf{R}^{N-1} = \{u = (0, u_2, \cdots, u_N)^t \in \mathbf{R}^N\}$ be a subspace of $\mathbf{R}^N$ and $G_{\mathbf{R}}(n, N-1)$ be a submanifold of $G_{\mathbf{R}}(n,N)$ generated by the n-dimensional subspace of $\mathbf{R}^{N-1}$. Let $G_{\mathbf{R}}(n-1, N-1)$ be a submanifold of $G_{\mathbf{R}}(n,N)$ such that every element of $G_{\mathbf{R}}(n-1,N-1)$ contains the vector $\tilde{e}_1 = (1, 0, \cdots, 0)^t \in \mathbf{R}^N$. It is easy to see that $f^{-1}(0) = G_{\mathbf{R}}(n, N-1)$ and $f^{-1}(1) = G_{\mathbf{R}}(n-1, N-1)$.

Now we show that the critical submanifolds $f^{-1}(0)$ and $f^{-1}(1)$ of f are non-degenerate and compute their indices.

On $f^{-1}(0) = G_{\mathbf{R}}(n, N-1)$, $x_{i1} = 0$, $i = 1, \cdots, n$, $\tilde{e}_{n+1} = (1, 0, \cdots, 0)^t$, then the tangent space of $f^{-1}(0)$ is generated by

$$e_i e_\alpha^t + e_\alpha e_i^t, \quad \alpha \neq n+1.$$

On $f^{-1}(1) = G_{\mathbf{R}}(n-1, N-1)$, $\tilde{e}_1 = (1, 0, \cdots, 0)$, $G_{\mathbf{R}}(n-1, N-1)$ is generated by $\tilde{e}_1, e_2, \cdots, e_n$, then the tangent space of $f^{-1}(1)$ is generated by

$$e_i e_\alpha^t + e_\alpha e_i^t, \quad i \neq 1.$$

By simple computation, on the critical submanifolds, we have

$$d^2 f = -\sum \omega_j^\alpha \omega_i^\alpha \langle e_j e_i^t + e_i e_j^t, E_{11}\rangle + \sum \omega_i^\alpha \omega_i^\beta \langle e_\alpha e_\beta^t + e_\beta e_\alpha^t, E_{11}\rangle.$$

Then

$$d^2 f|_{f^{-1}(0)} = 2\sum \omega_i^{n+1}\omega_i^{n+1}\langle e_{n+1}e_{n+1}^t, E_{11}\rangle = 2\sum \omega_i^{n+1}\omega_i^{n+1},$$

$$d^2 f|_{f^{-1}(1)} = -2\sum \omega_1^\alpha \omega_1^\alpha \langle e_1 e_1^t, E_{11}\rangle = -2\sum \omega_1^\alpha \omega_1^\alpha.$$

By Lemma 2.2, the critical submanifolds of f are all non-degenerate. These complete the proof of the theorem. □

By Morse theory, it can be shown that every differentiable manifold has the homotopy type of a CW complex. As in [4] or [5], let

$$0 \leq a_0 \leq a_1 \leq \cdots \leq a_n \leq N - n$$

be a sequence of integers. These give a CW complex structure on the Grassmann manifold $G_{\mathbf{F}}(n,N)$. For every such $(a_0, a_1, \cdots, a_n)$, there is one cell of dimension $c(a_0 + a_1 + \cdots + a_n)$. The homologies of the Grassmann manifold can be computed by means of the Schubert varieties (see [4] or [5]). There is a close relation between Theorem 3.1 and the Schubert varieties:

The elements $(a_0, a_1, \cdots, a_n)$ can be divided into two classes:

(1) $(a_0, a_1, \cdots, a_n)$, where $a_n \leq N - n - 1$;

(2) $(a_0, a_1, \cdots, a_{n-1}, N-n)$, where $a_{n-1} \leq N - n$.

Let $\{\rho_1, \rho_2, \rho_3\}$ be a partition of unity on $[0,1]$ such that $\mathrm{supp}(\rho_1) \subset [0, \frac{1}{4}]$, $\mathrm{supp}(\rho_2) \subset [\frac{1}{8}, \frac{7}{8}]$, $\mathrm{supp}(\rho_3) \subset [\frac{3}{4}, 1]$ and $\frac{d\rho_1}{dt} \leq 0$, $\frac{d\rho_3}{dt} \geq 0$. Let $h_1 \leq 0, h_2 \geq 0$ be two non-degenerate Morse functions on $f^{-1}(0)$ and $f^{-1}(1)$ respectively. The functions h_1, h_2 can be viewed as functions on neighborhoods of $f^{-1}(0), f^{-1}(1)$ in $G_{\mathbf{F}}(n,N)$ respectively, they are constants on the trajectories of $\mathrm{grad}(f)$. Define a function

$$\tilde{f} = \tilde{\rho}_1(h_1 + f) + \tilde{\rho}_2 f + \tilde{\rho}_3(h_2 + f) = \tilde{\rho}_1 h_1 + \tilde{\rho}_3 h_2 + f,$$

where $\tilde{\rho}_i = \rho_i \circ f,\ i = 1,2,3$.

Theorem 3.2. *$\tilde{f} : G_{\mathbf{F}}(n,N) \to \mathbf{R}$ is a non-degenerate Morse function and the critical points are that of h_1 and h_2. If p is a critical point of h_2 with index k, then the index of p is $k + c(N-n)$ with respect to the function $\tilde{f}$; if q is a critical point of h_1, then the indices of q with respect to the functions h_1 and $\tilde{f}$ are the same.*

For the proof, see [12].

When $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$, we can choice perfect Morse functions h_1, h_2 on $G_{\mathbf{F}}(n, N-1)$ and $G_{\mathbf{F}}(n-1, N-1)$ respectively. Then $\tilde{f}$ is also a perfect Morse function. Let $P_t(M)$ be the Poincaré polynomial for a manifold M.

Corollary 3.3. *For $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$, the Poincaré polynomial of $G_{\mathbf{F}}(n,N)$ can be represented by*

$$P_t(G_{\mathbf{F}}(n,N)) = P_t(G_{\mathbf{F}}(n,N-1)) + t^{c(N-n)} P_t(G_{\mathbf{F}}(n-1,N-1)).$$

For example, by simple computation, we have

$$P_t(G_{\mathbf{C}}(1,N)) = 1 + t^2 + t^4 + \cdots + t^{2(N-1)},$$

$$P_t(G_{\mathbf{C}}(2,5)) = 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12},$$

$$P_t(G_{\mathbf{C}}(2,7)) = 1 + t^2 + 2t^4 + 2t^6 + 3t^8 + 3t^{10} + 3t^{12} + 2t^{14} + 2t^{16} + t^{18} + t^{20},$$

$$P_t(G_{\mathbf{C}}(2,8)) = P_t(G_{\mathbf{C}}(2,7)) + t^{12} P_t(G_{\mathbf{C}}(1,7)),$$

$$P_t(G_{\mathbf{C}}(3,7)) = (1 + t^6) P_t(G_{\mathbf{C}}(2,5)) + t^8 P_t(G_{\mathbf{C}}(2,6)),$$

$$P_t(G_{\mathbf{C}}(5,10)) = (1 + t^{10})[t^{20} P_t(G_{\mathbf{C}}(2,7)) + (1 + t^8 + t^{10}) P_t(G_{\mathbf{C}}(3,7))].$$

By $P_t(G_{\mathbf{F}}(n,N)) = P_t(G_{\mathbf{F}}(N-n,N))$, we have

$$(t^{cn} - 1) P_t(G_{\mathbf{F}}(n,N-1)) = (t^{c(N-n)} - 1) P_t(G_{\mathbf{F}}(n-1,N-1)).$$

Now we study the trajectories of gradient vector field of the degenerate Morse function $f : G_{\mathbf{F}}(n,N) \to \mathbf{R}$ defined in Theorem 3.1. The gradient of the function $f : G_{\mathbf{F}}(n,N) \to \mathbf{R}$ is

$$\mathrm{grad}(f) = \frac{1}{2} \sum_{\tau} \langle \xi_\tau, E_{11} \rangle \xi_\tau,$$

where the tangent vectors ξ_τ are defined as in Lemma 2.1. For any $\pi \in G_{\mathbf{F}}(n,N) - (f^{-1}(0)\cup f^{-1}(1))$, let $e_i = (x_{i1}, x_{i2}, \cdots, x_{iN})^t$, $i = 1, \cdots, n$, be an orthonormal basis of π such that $0 < x_{11} < 1$, $x_{i1} = 0$ for $i > 1$. Denote $e_1(t) = (\cos t, (\sin t)x)^t$, where $x = (x_{12}, \cdots, x_{1N})/\sqrt{|x_{12}|^2 + \cdots + |x_{1N}|^2}$. Then there is $t_0 \in (0, \frac{\pi}{2})$ such that $e_1(t_0) = e_1$. Let $\gamma(t)$ be a curve in $G_{\mathbf{F}}(n,N)$ generated by orthonormal vectors $e_1(t), e_2, \cdots, e_n$. Then

$$f(\gamma(t)) = \cos^2 t, \quad \frac{df(\gamma(t))}{dt} = -\sin 2t,$$

$$\gamma(0) \in f^{-1}(1) = G_{\mathbf{F}}(n-1, N-1),\ \gamma(\frac{\pi}{2}) \in f^{-1}(0) = G_{\mathbf{F}}(n, N-1).$$

Note that $\dim \gamma(0)\cap\gamma(\frac{\pi}{2}) = c(n-1)$. Along the curve $\gamma(t)$, let

$$e_{n+1}(t) = (-\sin t, (\cos t)x)^t, \quad e_\alpha = (0, x_{\alpha 2}, \cdots, x_{\alpha N})^t, \quad \alpha > n+1,$$

be orthonormal complement of the vectors $e_1(t), e_2, \cdots, e_n$ in $\mathbf{F}^N$. Therefore

$$\mathrm{grad}(f)|_\gamma = -\frac{1}{2}\sin 2t(e_{n+1}(t)\bar{e}_1^t(t) + e_1(t)\bar{e}_{n+1}^t(t)) = -\frac{1}{2}\sin 2t\frac{d\gamma}{dt}.$$

This shows that the curve γ is a trajectory of the vector field $\mathrm{grad}(f)$ on $G_{\mathbf{F}}(n,N)$.

It is also easy to see that the vector $\frac{d\gamma}{dt}(0)$ is normal to $f^{-1}(1)$ and the vector $\frac{d\gamma}{dt}(\frac{\pi}{2})$ is normal to $f^{-1}(0)$. Let $\mathbf{F}P^{N-n-1} = G_{\mathbf{F}}(1, N-n)$ be a subspace of $f^{-1}(0)$ such that $e_2, \cdots, e_n \in \pi$ for any $\pi \in \mathbf{F}P^{N-n-1}$. Let $\mathbf{F}P^{n-1} = G_{\mathbf{F}}(1, n-1)$ be a subspace of $f^{-1}(1)$, any $\pi \in \mathbf{F}P^{n-1}$ be generated by $e_1 = (1, 0, \cdots, 0)^t, \tilde{e}_2, \cdots, \tilde{e}_n$, where $\tilde{e}_2, \cdots, \tilde{e}_n \in \gamma(\frac{\pi}{2})$.

Theorem 3.4. *The trajectories of grad(f) give the maps from $\mathbf{F}P^{N-n-1}$ to $\gamma(0)$ and $\mathbf{F}P^{n-1}$ to $\gamma(\frac{\pi}{2})$ respectively.*

When $n = 1$, these gives the following canonical cell decomposition of the projective space $\mathbf{F}P^{N-1}$

$$\mathbf{F}P^0 \subset \mathbf{F}P^1 \subset \cdots \subset \mathbf{F}P^{N-2} \subset \mathbf{F}P^{N-1}.$$

In the following we assume $n, N-n \geq 2$.

Theorem 3.5. *Let $g:\ G_{\mathbf{F}}(n,N) \to \mathbf{R},\ g(\pi) = \langle\varphi(\pi), E_{12}\rangle$. The function g is a degenerate Morse function with critical submanifolds $g^{-1}(0) = G_{\mathbf{F}}(n-2, N-2)\bigcup G_{\mathbf{F}}(n, N-2)$, $g^{-1}(-1) = \widetilde{G}_{\mathbf{F}}(n-1, N-2)$, $g^{-1}(1) = G_{\mathbf{F}}(n-1, N-2)$. The indices on $G_{\mathbf{F}}(n-2, N-2), G_{\mathbf{F}}(n, N-2)$, $\widetilde{G}_{\mathbf{F}}(n-1, N-2)$, $G_{\mathbf{F}}(n-1, N-2)$ are $c(N-n), cn, 0, c(N-1)$ respectively.*

Proof. Let $e_i = (x_{i1}, x_{i2}, \cdots, x_{iN})^t$, then,

$$g(\pi) = \mathrm{Re}\ \sum_i (x_{i1}\bar{x}_{i2} + x_{i2}\bar{x}_{i1}) = 2\mathrm{Re}\ \sum_i x_{i1}\bar{x}_{i2}.$$

We can assume $x_{i1} = 0$ for $i > 1$ and x_{11} a real number, this shows $-1 \leq g(\pi) \leq 1$. The critical points of function g are determined by

$$dg = \sum_{i,\alpha} \langle e_\alpha \omega_i^\alpha \bar{e}_i^t + e_i \bar{\omega}_i^\alpha \bar{e}_\alpha^t, E_{12} \rangle = 0,$$

where $\varphi(\pi) = \sum e_i \bar{e}_i^t$.

We prove the theorem for the case of $\mathbf{F} = \mathbf{H}$. $dg = 0$ if and only if

$$\langle e_\alpha \bar{e}_i^t + e_i \bar{e}_\alpha^t, E_{12} \rangle = 0, \quad \langle e_\alpha i \bar{e}_i^t - e_i i \bar{e}_\alpha^t, E_{12} \rangle = 0,$$

$$\langle e_\alpha j \bar{e}_i^t - e_i j \bar{e}_\alpha^t, E_{12} \rangle = 0, \quad \langle e_\alpha k \bar{e}_i^t - e_i k \bar{e}_\alpha^t, E_{12} \rangle = 0,$$

for any i, α. Obviously, we have

$$\begin{aligned} &\langle e_\alpha \bar{e}_i^t + e_i \bar{e}_\alpha^t, E_{12} \rangle \\ &= \mathrm{Re}\ (x_{\alpha 1} \bar{x}_{i2} + x_{\alpha 2} \bar{x}_{i1} + x_{i1} \bar{x}_{\alpha 2} + x_{i2} \bar{x}_{\alpha 1}) \\ &= 2\mathrm{Re}\ [x_{\alpha 1} \bar{x}_{i2} + x_{\alpha 2} \bar{x}_{i1}], \end{aligned}$$

and

$$\begin{aligned} &\langle e_\alpha i \bar{e}_i^t - e_i i \bar{e}_\alpha^t, E_{12} \rangle \\ &= \mathrm{Re}\ [(x_{\alpha 1} i \bar{x}_{i2} - x_{i2} i \bar{x}_{\alpha 1}) + (x_{\alpha 2} i \bar{x}_{i1} - x_{i1} i \bar{x}_{\alpha 2})] \\ &= 2\mathrm{Re}\ [x_{\alpha 1} i \bar{x}_{i2} + x_{\alpha 2} i \bar{x}_{i1}]. \end{aligned}$$

Similarly,

$$\langle e_\alpha j \bar{e}_i^t - e_i j \bar{e}_\alpha^t, E_{12} \rangle = 2\mathrm{Re}\ [x_{\alpha 1} j \bar{x}_{i2} + x_{\alpha 2} j \bar{x}_{i1}],$$

$$\langle e_\alpha k \bar{e}_i^t - e_i k \bar{e}_\alpha^t, E_{12} \rangle = 2\mathrm{Re}\ [x_{\alpha 1} k \bar{x}_{i2} + x_{\alpha 2} k \bar{x}_{i1}].$$

For any $u, v \in \mathbf{H}, a \in \mathrm{Im}\mathbf{H}$, the following holds

$$\mathrm{Re}\ (u a \bar{v}) = \mathrm{Re}\ (-u \overline{va}) = \mathrm{Re}\ (-\bar{u} v a).$$

These show π is a critical point of g if and only if

$$x_{\alpha 1} \bar{x}_{i2} + x_{\alpha 2} \bar{x}_{i_1} = 0, \quad \text{for all } i, \alpha.$$

On the critical submanifolds, we have

$$d^2 g = -\sum_{i,j,\alpha} \langle e_j \bar{\omega}_j^\alpha \omega_i^\alpha \bar{e}_i^t + e_i \bar{\omega}_i^\alpha \omega_j^\alpha \bar{e}_j^t, E_{12} \rangle + \sum_{i,j,\alpha} \langle e_\alpha \omega_i^\alpha \bar{\omega}_i^\beta \bar{e}_\beta^t + e_\beta \omega_i^\beta \bar{\omega}_i^\alpha \bar{e}_\alpha^t, E_{12} \rangle.$$

(1) Let $\mathbf{H}^{N-2} = \{(0, 0, x_3, \cdots, x_N)^t \in \mathbf{H}^N\}$ be a subspace of $\mathbf{H}^N$ and $G_{\mathbf{H}}(n, N-2) = \{\pi \in G_{\mathbf{H}}(n, N) \mid \pi \subset \mathbf{H}^{N-2}\}$ be submanifold of $G_{\mathbf{H}}(n, N)$. Then $G_{\mathbf{H}}(n, N-2)$ is a critical submanifold of function g and $g|_{G_{\mathbf{H}}(n,N-2)} \equiv 0$. In this case, $x_{i1} = x_{i2} = 0$ for $i = 1, \cdots, n$, we can assume

$$e_{n+1} = (1, 0, 0, \cdots, 0)^t, \ e_{n+2} = (0, 1, 0, \cdots, 0)^t.$$

Then

$$\begin{aligned}
&d^2g|_{G_{\mathbf{H}}(n,N-2)}\\
&= 2\mathrm{Re}\ \sum_i(\omega_i^{n+1}\bar{\omega}_i^{n+2}+\omega_i^{n+2}\bar{\omega}_i^{n+1})\\
&= 2\sum_i[(a_i^{n+1}+a_i^{n+2})^2-(a_i^{n+1}-a_i^{n+2})^2+(b_i^{n+1}+b_i^{n+2})^2\\
&\quad -(b_i^{n+1}-b_i^{n+2})^2+(c_i^{n+1}+c_i^{n+2})^2-(c_i^{n+1}-c_i^{n+2})^2\\
&\quad +(d_i^{n+1}+d_i^{n+2})^2-(d_i^{n+1}-d_i^{n+2})^2],
\end{aligned}$$

where $\omega_i^\alpha = a_i^\alpha+ib_i^\alpha+jc_i^\alpha+kd_i^\alpha$. As in the proof of Theorem 3.1, we can show that the critical submanifold $G_{\mathbf{H}}(n,N-2)$ is non-degenerate with index $4n$.

(2) Let $\tilde{e}_1=(1,0,\cdots,0)^t$, $\tilde{e}_2=(0,1,0,\cdots,0)^t\in\mathbf{H}^N$ and $G_{\mathbf{H}}(n-2,N-2)=\{\pi\in G_{\mathbf{H}}(n,N)\mid \tilde{e}_1,\tilde{e}_2\in\pi\}$ be a submanifold of $G_{\mathbf{H}}(n,N)$. Then we have $x_{\alpha 1}=x_{\alpha 2}=0$, $\alpha=n+1,\cdots,N$, for any $\pi\in G_{\mathbf{H}}(n-2,N-2)$. Therefore $g|_{G_{\mathbf{H}}(n-2,N-2)}\equiv 0$ and $G_{\mathbf{H}}(n-2,N-2)$ is a critical submanifold of g,

$$d^2g|_{G_{\mathbf{H}}(n-2,N-2)} = -2\mathrm{Re}\ (\sum_\alpha \bar{\omega}_1^\alpha\omega_2^\alpha+\bar{\omega}_2^\alpha\omega_1^\alpha).$$

The critical submanifold $G_{\mathbf{H}}(n-2,N-2)$ is non-degenerate with index $4(N-n)$.

(3) Now we study the case of the numbers x_{i1},x_{i2} are not all zeros and so are the numbers $x_{\alpha 1},x_{\alpha 2}$. Assuming $x_{i1}=0$ for $i>1$, $x_{j2}=0$ for $j>2$; $x_{\alpha 1}=0$ for $\alpha>n+1$, $x_{\beta 2}=0$ for $\beta>n+2$. Then the conditions $x_{\alpha 1}\bar{x}_{i2}+x_{\alpha 2}\bar{x}_{i_1}=0$ become

$$\frac{\bar{x}_{11}}{\bar{x}_{12}}=\frac{0}{\bar{x}_{22}}=-\frac{x_{n+1\ 1}}{x_{n+1\ 2}}=-\frac{0}{x_{n+2\ 2}}.$$

If $x_{11}=0$, we can assume $x_{22}=0$. Then if $x_{11}=0$, we have $x_{12}=0$. Therefore $x_{11}\neq 0, x_{12}\neq 0$ in this case. Similarly, $x_{n+1\ 1}\neq 0, x_{n+1\ 2}\neq 0$. Hence $x_{22}=x_{n+2\ 2}=0$. By

$$\frac{\bar{x}_{11}}{\bar{x}_{12}}=-\frac{x_{n+1\ 1}}{x_{n+1\ 2}},\quad |x_{11}|^2+|x_{n+1\ 1}|^2=1,\ |x_{12}|^2+|x_{n+1\ 2}|^2=1,$$

and the vectors $e_1\perp e_{n+1}$, we have

$$e_1=(\frac{1}{\sqrt{2}},\pm\frac{1}{\sqrt{2}},0,\cdots,0)^t,\quad e_i=(0,0,x_{i3},\cdots,x_{iN})^t,\quad i>1.$$

Let $G_{\mathbf{H}}(n-1,N-2)$ be the subset of $\pi\in G_{\mathbf{H}}(n,N)$ which is generated by $e_1=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},0,\cdots,0)^t$, $e_i=(0,0,x_{i3},\cdots,x_{iN})^t$, $i>1$. Similarly, Let $\widetilde{G}_{\mathbf{H}}(n-1,N-2)\subset G_{\mathbf{H}}(n,N)$ be the subset of π which is generated by $\tilde{e}_1=(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},0,\cdots,0)^t$, $e_i=(0,0,x_{i3},\cdots,x_{iN})^t$, $i>1$. By construction,

$G_{\mathbf{H}}(n-1, N-2)$ and $\widetilde{G}_{\mathbf{H}}(n-1, N-2)$ are critical submanifolds of function g,

$$g|_{G_{\mathbf{H}}(n-1,N-2)} \equiv 1, \quad g|_{\widetilde{G}_{\mathbf{H}}(n-1,N-2)} \equiv -1.$$

By our assumption, $g(\pi) = 2\mathrm{Re}\ (x_{11}\bar{x}_{12})$ and $|x_{11}|^2 + |x_{12}|^2 \leq 1$, this shows

$$g^{-1}(1) = G_{\mathbf{H}}(n-1, N-2), \quad g^{-1}(-1) = \widetilde{G}_{\mathbf{H}}(n-1, N-2).$$

On $G_{\mathbf{H}}(n-1, N-2)$, we can set $e_{n+1} = (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \cdots, 0)^t$; on $\widetilde{G}_{\mathbf{H}}(n-1, N-2)$, we can set $\tilde{e}_{n+1} = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, \cdots, 0)^t$. Then we have

$$d^2 g|_{g^{-1}(-1)} = \mathrm{Re}\ [\sum_{\alpha} \bar{\omega}_1^{\alpha}\omega_1^{\alpha} + \sum_i \omega_i^{n+1}\bar{\omega}_i^{n+1}],$$

$$d^2 g|_{g^{-1}(1)} = -\mathrm{Re}\ [\sum_{\alpha} \bar{\omega}_1^{\alpha}\omega_1^{\alpha} + \sum_i \omega_i^{n+1}\bar{\omega}_i^{n+1}].$$

As in the proof of Theorem 3.1, we can show that the critical submanifolds $g^{-1}(-1)$ and $g^{-1}(1)$ are non-degenerate with indices 0 and $4(N-1)$ respectively. $\square$

As Corollary 3.3, we have

Corollary 3.6. *For $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$, the Poincaré polynomial of $G_{\mathbf{F}}(n, N)$ can be represented by*

$$\begin{aligned} P_t(G_{\mathbf{F}}(n,N)) = \quad & t^{cn}P_t(G_{\mathbf{F}}(n, N-2)) + t^{c(N-n)}P_t(G_{\mathbf{F}}(n-2, N-2)) \\ & +(1+t^{c(N-1)})P_t(G_{\mathbf{F}}(n-1, N-2)), \end{aligned}$$

where $n, N-n \geq 2$.

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