

INTRINSIC SYMMETRY GROUPS OF LINKS WITH 8 AND FEWER CROSSINGS

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ABSTRACT. We present new computations of “intrinsic” symmetry groups for knots and links of 8 or fewer crossings. The standard symmetry group for a link is the mapping class group $MCG(S^3, L)$ or $Sym(L)$ of the pair (S^3, L) . Elements in this symmetry group can (and often do) fix the link and act nontrivially only on its complement. We ignore such elements and focus on the “intrinsic” symmetry group of a link, defined to be the image $\Sigma(L)$ of the natural homomorphism $MCG(S^3, L) \rightarrow MCG(S^3) \times MCG(L)$. This different symmetry group, first defined by Whitten in 1969, records directly whether L is isotopic to a link L' obtained from L by permuting components or reversing orientations. While the standard symmetry groups $Sym(L)$ have been computed for knots of up to 10 crossings and most links with up to 9 crossings, we give here the first comprehensive tables of the groups $\Sigma(L)$.

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1. INTRODUCTION

The symmetry group of a link L is defined to be the mapping class group $\text{MCG}(L)$ (or $\text{Sym}(L)$) of the pair (S^3, L) . The study of this symmetry group is a classical topic in knot theory, and these groups have now been computed for prime knots and links in several ways. Kodama and Sakuma [KS92] used a method in Bonahon and Siebenmann [BS09] to compute these groups for all but three of the knots of 10 and fewer crossings in 1992. In the same year, Weeks and Henry used the program SnapPea to compute the symmetry groups for hyperbolic knots and links of 9 and fewer crossings [HW92]. These efforts followed earlier tabulations of symmetry groups by Boileau and Zimmermann [BZ87], who found symmetry groups for nonelliptic Montesinos links with 11 or fewer crossings.

The goal of this paper is to compute a different group of symmetries of a link L given by the image of the natural homomorphism

$$\pi : \text{Sym}(L) = \text{MCG}(S^3, L) \rightarrow \text{MCG}(S^3) \times \text{MCG}(L).$$

Since these symmetries record an action on L itself (and only record the orientation of the ambient S^3), we will call them “intrinsic” symmetries of L to distinguish them from the standard symmetry group. As we will show below, the group on the right was defined (though not in this way) by Whitten in 1969 [Whi69], following ideas of Fox. They denoted this group $\Gamma(L)$ or Γ_μ , where μ is the number of components of L . We can write this group as a semidirect product of \mathbf{Z}_2 groups encoding the orientation of each component of the link L with the permutation group S_μ exchanging components of L (cf. Definition 2.1), finally crossed with another \mathbf{Z}_2 recording the orientation of S^3 :

$$\Gamma(L) = \Gamma_\mu = \mathbf{Z}_2 \times (\mathbf{Z}_2^\mu \rtimes S_\mu).$$

It is clear that an element $\gamma = (\epsilon_0, \epsilon_1, \dots, \epsilon_\mu, p) \in \Gamma(L)$ acts on L to produce a new link L^γ . If $\epsilon_0 = +1$, then L^γ and L are the same as sets (but the components of L have been renumbered and reoriented), while if $\epsilon_0 = -1$ the new link L^γ is the mirror image of L (again with renumbering and reorientation). We can then define the symmetry subgroup $\Sigma(L)$ by

$$\gamma \in \Sigma(L) \iff \text{there is an isotopy from } L \text{ to } L^\gamma \text{ preserving component numbering and orientation.}$$

For knots, $\Sigma(L) < \Gamma_1 = \mathbf{Z}_2 \times \mathbf{Z}_2$. Here the five subgroups of $\mathbf{Z}_2 \times \mathbf{Z}_2$ correspond to the standard descriptions of the possible symmetries for a knot, as shown in Table 1.

Symmetry subgroup of Γ_1	Name	Example(s)
$\{(1, 1)\}$	No symmetry	$9_{32}, 9_{33}$
$\{(1, 1), (-1, 1)\}$	(+) amphichiral symmetry	12_{427}
$\{(1, 1), (1, -1)\}$	invertible ¹ symmetry	3_1
$\{(1, 1), (-1, -1)\}$	(-) amphichiral ² symmetry	8_{17}
Γ_1	full symmetry	4_1

TABLE 1. The five standard symmetry types for knots correspond to the five subgroups of the Whitten group Γ_1 .

¹Conway [Con70] calls this *reversible* symmetry.

²Conway [Con70] calls this *invertible* symmetry.

For links, the situation is more interesting, as the group $\Gamma(L)$ is more complicated. In the case of two-component links, the group $\Gamma_2 = \mathbf{Z}_2 \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \times S_2)$ is a nonabelian 16 element group isomorphic to $\mathbf{Z}_2 \times D_4$. The various subgroups of Γ_2 do not all have standard names, but we will call a link *purely invertible* if $(1, -1, \dots, -1, e) \in \Sigma(L)$, and say that components (i, j) have a *pure exchange* symmetry if $(1, 1, \dots, 1, (ij)) \in \Sigma(L)$. For two-component links, we will say that L has pure exchange symmetry if its two components have that symmetry. The question of which links have this symmetry goes back at least to Fox's 1962 problem list in knot theory [Fox62, Problem 11]³. For example, 2_1^2 (the Hopf link) has pure exchange symmetry while we will show that 7_4^2 does not.

We similarly focus on the pure invertibility symmetry: we show that 45 out of the 47 prime links with 8 crossings or less are purely invertible. The two exceptions are 6_2^3 (the Borromean rings) and 8_5^3 ; we show that both of these are invertible using some nontrivial permutation, i.e., $(1, -1, -1, -1, p) \in \Sigma(L)$ for some $p \neq e$. (Whitten found examples of more complicated links which are not invertible even when a nontrivial permutation is allowed [Whi71].)

Increasing the number of components in a link greatly increases the number of possible types of symmetry. Table 2 lists the number of subgroups of Γ_μ ; each different subgroup represents a different intrinsic symmetry group that a μ -component link might have. We note that if $\Sigma(L)$ is the symmetry subgroup of link L , then the symmetry subgroup of L^γ is the conjugate subgroup $\Sigma(L^\gamma) = \gamma\Sigma(L)\gamma^{-1}$. Therefore, it suffices to only examine the number of mutually nonconjugate subgroups of Γ_μ in order to specify all of the different intrinsic symmetry groups. Table 2 also lists the number of conjugacy classes of subgroups of Γ_μ , and the number of these which appear for prime links of 8 or fewer crossings.

μ	$ \Gamma_\mu $	# subgroups	# subgroups (up to conjugacy)	nonconjugate ones for ≤ 8 crossings
1	4	5	5	3
2	16	35	27	5
3	96	420	131	7
4	768	9417	994	3
5	7680	270131	6382	0

TABLE 2. The number of subgroups of Γ_μ ; each one represents a different intrinsic symmetry group possible for a μ -component link. The number of nonconjugate symmetries (the fourth column) is given by the number of conjugacy classes of subgroups.. This article computes the symmetry group for all prime links of 8 or fewer crossings; the last column summarizes our results.

Why are the $\Sigma(L)$ groups interesting? First, it is often more natural to consider the restricted group $\Sigma(L)$ than the generally larger $\text{Sym}(L)$. Sakuma [Sak89] has shown that a knot K has a finite symmetry group if and only if K is a hyperbolic knot, a torus knot, or a cable of a torus knot. Thus for many⁴ knots, the group $\text{Sym}(L)$ contains infinitely many elements which act nontrivially on the complement of L but fix the knot itself. We ignore such elements, which lie in the kernel of the natural homomorphism $\pi : \text{MCG}(S^3, L) \rightarrow \text{MCG}(S^3) \times \text{MCG}(L)$. In fact, even when $\text{MCG}(S^3, L)$ is finite, we give various examples below where π has nontrivial kernel. It is often difficult to describe an element of $\text{Sym}(L)$ in $\ker \pi$ explicitly, but it is always simple to understand the exact meaning of the statement $\gamma \in \Sigma(L)$.

As an application, if one is interested in classifying knots and links up to oriented, labeled ambient isotopy, it is important to know the symmetry group $\Sigma(L)$ for each prime link type L , since links related

³Actually, Fox was willing to call L *interchangeable* if $\Sigma(L)$ contained any element in the form $(\epsilon_0, \epsilon_1, \epsilon_2, (12))$, but we find it more useful to focus on the “pure exchange” symmetry.

⁴Sakuma gives as an explicit example the untwisted double of a two-bridge knot.

Crossings	Number of link types by components and crossings									
	1-U	1-OS	2-U	2-OS	3-U	3-OS	4-U	4-OS	All Links-U	All Links-OS
0	1	1	1	1					1	1
2			1	2					1	2
3	1	2								
4	1	1	1	4					1	4
5	2	4	1	2					1	2
6	3	5	3	10	3	18			6	44
7	7	14	8	38	1	8			9	40
8	21	38	16	78	10	200	3	120	29	398

TABLE 3. This table shows the number of distinct isotopy classes of links by crossing number and number of components in the link. The columns labeled “ n -U” count link types for n -component links in the usual way, where the components are unlabeled and unoriented, and the mirror image of a given link is considered to have the same link type, regardless of whether the two are ambient isotopic. The columns labeled “ n -OS” give finer information, considering two n component links to have the same type if and only if they are oriented, labeled ambient isotopic. As we can see, this stricter definition leads to a much larger collection of link types.

by an element in $\Gamma(L)$ outside $\Sigma(L)$ are not (oriented, labeled) ambient isotopic. The number of different links related by an element of $\Gamma(L)$ to a given link of prime link type L is given by the number of cosets of $\Sigma(L)$ in $\Gamma(L)$. If we count these cosets instead of prime link types, the number of actual knots and link types of a given crossing number is actually quite a bit larger than the usual table of prime knot and link types suggests. (See Table 3.)

Second, $\Sigma(L)$ seems likely to be eventually relevant in applications. For instance, when studying DNA links, each loop of the link has generally has a unique sequence of base pairs which provide an orientation and an unambiguous labeling of each component of the link. In such a case, the question of whether two components in a link can be interchanged may prove to be of real significance.

Last, we are interested in the topic of tabulating composite knots and links. Since the connect sum of different symmetry versions of the same knot type can produce different knots (such as the square knot, which is the connect sum of a trefoil and its mirror image, and the granny knot, which is the connect sum of two trefoils with the same handedness), keeping track of the action of $\Sigma(L)$ is a crucial element in this calculation. We treat this topic in a forthcoming manuscript [CMP11].

This paper provides a hands-on calculation of the groups $\Sigma(L)$ for all links of 8 and fewer crossings. As far as we know, this is the first comprehensive list of these $\Sigma(L)$ groups, though Hillman [Hil86] provides examples of various two-component links (including some split links) with symmetry groups equal to 12 different subgroups of Γ_2 .

We started from the excellent table of oriented alternating links provided by Cerf [Cer98]. Cerf considers the effect of reversing orientations of components of her links, but does not compute the effect of permutations. Our more detailed calculation does not depend on this information, so our paper also provides a check on Cerf’s symmetry calculations. For alternating links, we have kept component numbers and orientations consistent with her table. For nonalternating links, we use component numbers and orientations consistent with those of Doll and Hoste [DH91]. Like Cerf, Doll and Hoste considered the effect of reversing the orientation of individual components of these links, but not the effect of permutations of components. We note that these component numbers and orientations are *not* consistent with those used in SnapPea. The

component numbers and orientations in `SnapPea` seem to have been chosen arbitrarily sometime in the 1980s by Joe Christy when he digitized the Rolfsen table [CD10].

We note that it is possible to calculate $\Sigma(L)$ directly from $\text{Sym}(L)$ for certain classes of links (cf. Exercise 16.12 in [BS09] for algebraic links). For hyperbolic links, it is known that the symmetries of L are realized by isometries of the complete hyperbolic structure on $S^3 - L$, and Weeks and Henry computed these isometries by showing that they are combinatorial symmetries of a canonical ideal triangulation of the complement [HW92, Wee93] and constructing these triangulations and symmetries using the computer program `SnapPea`. We used `SnapPea` to compute $\text{Sym}(L)$ for all hyperbolic links of 8 and fewer crossings. Then, using the Python interface provided by `SnapPy` we explicitly computed the image of $\pi : \text{MCG}(S^3, L) \rightarrow \text{MCG}(S^3) \times \text{MCG}(L)$ to verify our calculations. The results agreed⁵ with the tables we give below, meaning that our results serve as an independent verification of `SnapPea`'s accuracy for these links. While recomputing our data in this way was interesting, it does not tell the whole story. For instance, `SnapPea`'s results do not give any insight into how to explicitly construct the (often complicated) isotopies required to actually realize a given element of $\Sigma(L)$.

Finally, some notational comments: we henceforth use the term ‘link’ to refer to a prime, multicomponent link unless otherwise indicated. We use the Rolfsen notation for links but provide the Thistlethwaite notation as well in our summary tables (Tables 9, 11, and 12).

2. THE WHITTEN GROUP

We begin by giving the details of our construction of the Whitten group $\Gamma(L)$ and the symmetry group Σ . Consider operations on an oriented, labeled link L with μ components. We may reverse the orientation of any of the components of L or permute the components of L by any element of the permutation group S_n . However, these operations must interact with each as well: if we reverse component 3 and exchange components 3 and 5, we must decide whether the orientation is reversed before or after the permutation. Further, we can reverse the orientation on the ambient S^3 as well, a process which is clearly unaffected by the permutation. To formalize our choices, we follow [Whi69] to introduce the Whitten group of an n component link.

Definition 2.1. Consider the homomorphism given by

$$\omega : S_\mu \longmapsto \text{Aut}(\mathbf{Z}_2^{\mu+1}), \quad p \longmapsto \omega(p)$$

where $\omega(p)$ is defined as

$$\omega(p)(\epsilon_0, \epsilon_1, \epsilon_2 \dots \epsilon_\mu) = (\epsilon_0, \epsilon_{p(1)}, \epsilon_{p(2)} \dots \epsilon_{p(\mu)}).$$

For $\gamma = (\epsilon_0, \epsilon_1, \dots, \epsilon_\mu, p)$, and $\gamma' = (\epsilon'_0, \epsilon'_1, \dots, \epsilon'_\mu, q) \in \mathbf{Z}_2^{\mu+1} \rtimes_\omega S_\mu$, we define the *Whitten group* Γ_μ as the semidirect product $\Gamma_\mu = \mathbf{Z}_2^{\mu+1} \rtimes_\omega S_\mu$ with the group operation

$$\begin{aligned} \gamma * \gamma' &= (\epsilon_0, \epsilon_1, \epsilon_2 \dots \epsilon_\mu, p) * (\epsilon'_0, \epsilon'_1, \epsilon'_2 \dots \epsilon'_\mu, q) \\ &= ((\epsilon_0, \epsilon_1, \epsilon_2 \dots \epsilon_\mu) \cdot \omega(p)(\epsilon'_0, \epsilon'_1, \epsilon'_2 \dots \epsilon'_\mu), qp) \\ &= (\epsilon_0 \epsilon'_0, \epsilon_1 \epsilon'_{p(1)}, \epsilon_2 \epsilon'_{p(2)} \dots \epsilon_\mu \epsilon'_{p(\mu)}, qp) \end{aligned}$$

We will also use the notation $\Gamma(L)$ to refer to the Whitten group Γ_μ .

⁵We had to redraw several links where the component numbering and orientation given by default in `SnapPea` differed from our choice of numbering and orientation to make the groups agree. The default link data in `SnapPea` results in several cases in subgroups conjugate to those we give below.

2.1. Link operations. Given a link L consisting of μ oriented knots in S^3 , we may order the knots and write

$$L = K_1 \cup K_2 \cup \cdots \cup K_\mu.$$

Consider the following operations on L :

- (1) Permuting the K_i .
- (2) Reversing the orientation of any set of K_i 's
- (3) Reversing the orientation on S^3 (mirroring L).

Let γ be a combination of any of the moves (1), (2), or (3). We think of $\gamma = (\epsilon_0, \epsilon_1, \dots, \epsilon_\mu, p)$ as an element of the set $\mathbf{Z}_2^{\mu+1} \times S_\mu$ in the following way. Let

$$\epsilon_0 = \begin{cases} -1, & \text{if } \gamma \text{ mirrors } L \\ +1, & \text{if } \gamma \text{ does not mirror } L \end{cases}$$

and

$$\epsilon_i = \begin{cases} -1, & \text{if } \gamma \text{ reverses the orientation of } K_{p(i)} \\ +1, & \text{if } \gamma \text{ does not reverse the orientation of } K_{p(i)} \end{cases}$$

Lastly, let $p \in S_\mu$ be the permutation of the K_i associated to γ . To be explicitly clear, permutation p permutes the labels of the components; the component originally labeled i will be labeled $p(i)$ after the action of γ .

For each element, γ in $\mathbf{Z}_2^{\mu+1} \times S_\mu$, we define

$$(1) \quad L^\gamma = \gamma(L) = \epsilon_1 K_{p(1)}^{(*)} \cup \epsilon_2 K_{p(2)}^{(*)} \cup \cdots \cup \epsilon_\mu K_{p(\mu)}^{(*)} = \bigcup_{i=1}^{\mu} \epsilon_i K_{p(i)}^{(*)}$$

where $-K_i$ is K_i with orientation reversed, K_i^* is the mirror image of K_i and the $(*)$ appears above if and only if $\epsilon_0 = -1$. Note that $\gamma(L)$ is also a link with μ components. It is worth stressing that we first permute the knots and then reverse the appropriate ones.

Example 2.2. Let $L = K_1 \cup K_2 \cup K_3$ and $\gamma = (1, 1, -1, 1, (123))$. Then, $\gamma(L) = K_2 \cup -K_3 \cup K_1$.

Example 2.3. Let $L = K_1 \cup K_2 \cup K_3 \cup K_4$ and $\gamma = (-1, 1, 1, -1, -1, (14)(23))$. Then, $\gamma(L) = (K_4^* \cup K_3^* \cup -K_2^* \cup -K_1^*)$. Since we have reversed the orientation on S^3 , note that $\gamma(L)$ will be the mirror image of L as well.

We now confirm that this operation defines a group action of the Whitten group $\Gamma(L)$ on the set of links obtained from L by such transformations.

Proposition 2.4. *The Whitten group $\Gamma(L)$ is isomorphic to the group $\text{MCG}(S^3) \times \text{MCG}(L)$.*

Proof. We know that L is a disjoint union of μ copies of S^1 denoted $L = K_1 \sqcup \cdots \sqcup K_\mu$. Further, the mapping class groups of S^1 and S^3 are both \mathbf{Z}_2 , where the elements ± 1 correspond to orientation preserving and reversing diffeomorphisms of S^1 and S^3 . In general, the mapping class group of μ disjoint copies of a space is the semidirect product of the individual mapping class groups with the permutation group S_μ . This means that $\text{MCG}(L) = \mathbf{Z}_2 \times \cdots \times \mathbf{Z}_2 \times S_\mu$ and the Whitten group $\Gamma(L)$ has a bijective map to $\text{MCG}(S^3) \times \text{MCG}(L)$.

It remains to show that the group operation $*$ in the Whitten group maps to the group operation (composition of maps) in $\text{MCG}(S^3) \times \text{MCG}(L)$. To do so, we introduce some notation. Let $\gamma = (\epsilon_0, \epsilon_1, \dots, \epsilon_\mu, p)$ and $\gamma' = (\epsilon'_0, \epsilon'_1, \dots, \epsilon'_\mu, q)$. We must show

$$L^{\gamma * \gamma'} = (\gamma \circ \gamma')(L) = \gamma(\gamma'(L))$$

where $\gamma * \gamma'$ is the operation of the Whitten Group, Γ_μ .

Then,

$$\gamma'(L) = \gamma' \left(\bigcup_{i=1}^{\mu} K_i \right) = \bigcup_{i=1}^{\mu} \epsilon'_i K_{q(i)} = \bigcup_{i=1}^{\mu} X_i,$$

where $X_i = \epsilon'_i K_{q(i)}$. Thus,

$$\gamma(\gamma'(L)) = \gamma \left(\bigcup_{i=1}^{\mu} \epsilon'_i K_{q(i)} \right) = \gamma \left(\bigcup_{i=1}^{\mu} X_i \right) = \bigcup_{j=1}^{\mu} \epsilon_i X_{p(i)}.$$

Note that $X_{p(i)} = \epsilon'_{p(i)} K_{q(p(i))}$, which implies

$$\gamma(\gamma'(L)) = \bigcup_{i=1}^{\mu} \epsilon_i (\epsilon'_{p(i)} K_{q(p(i))}).$$

Now, $\gamma * \gamma' = (\epsilon_0 \epsilon'_0, \epsilon_1 \epsilon'_{p(1)}, \epsilon_2 \epsilon'_{p(2)}, \dots, \epsilon_{\mu} \epsilon'_{p(\mu)}, qp)$ and acts on L as

$$\begin{aligned} L^{\gamma * \gamma'} &= \epsilon_1 \epsilon'_{p(1)} K_{qp(1)} \cup \epsilon_2 \epsilon'_{p(2)} K_{qp(2)} \cup \dots \cup \epsilon_{\mu} \epsilon'_{p(\mu)} K_{qp(\mu)} \\ L^{\gamma * \gamma'} &= \bigcup_{i=1}^{\mu} \epsilon_{p(i)} (\epsilon'_i K_{qp(i)}) = \gamma(\gamma'(L)) \end{aligned}$$

We have dropped the notation for mirroring throughout the proof, because the two links clearly agree in this regard. The element $\gamma * \gamma'$ preserves the orientation of S^3 if and only if $\epsilon_0 \epsilon'_0 = 1$, i.e., if either both or neither of γ and γ' mirror L . \square

We can now define the subgroup of $\Gamma(L)$ which corresponds to the symmetries of the link L .

Definition 2.5. Given a link, L and $\gamma \in \Gamma(L)$, we say that L admits γ when there exists an isotopy taking each component of L to the corresponding component of L^{γ} which respects the orientations of the components. We define as the *Whitten symmetry group* of L ,

$$\Sigma(L) := \{\gamma \in \Gamma(L) \mid L \text{ admits } \gamma\}.$$

The Whitten symmetry group $\Sigma(L)$ is a subgroup of Γ_{μ} , and its left cosets represent the different isotopy classes of links L^{γ} among all symmetries γ . By counting the number of cosets, we determine the number of (labeled, oriented) isotopy classes of a particular prime link.

Next, we provide a few examples of symmetry subgroups. Recall that the first Whitten group $\Gamma_1 = \mathbf{Z}_2 \times \mathbf{Z}_2$ has order four and that $\Gamma_2 = \mathbf{Z}_2 \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \rtimes S_2)$ is a nonabelian 16 element group.

Example 2.6. Let $L = 4_1$, the figure eight knot. Since $L \sim -L \sim L^* \sim -L^*$, we have $\Sigma(4_1) = \Gamma_1$, so the figure eight knot has *full symmetry*. There is only one coset of $\Sigma(4_1)$ and hence only one isotopy class of 4_1 knots.

Example 2.7. Let $L = 3_1$, a trefoil knot. It is well known that $L \sim -L$ and $L^* \sim -L^*$, but $L \not\sim L^*$, so we have $\Sigma(3_1) = \{(1, 1, e), (1, -1, e)\}$. This means that the two cosets of $\Sigma(3_1)$ are $\{(1, 1, e), (1, -1, e)\}$ and $\{(-1, -1, e), (-1, 1, e)\}$, and there are two isotopy classes of 3_1 knots. A trefoil knot is thus *invertible*.

Example 2.8. Let $L = 7_5^2$, whose components are an unknot K_1 and a trefoil K_2 . In section 5.2, we determine all symmetry groups for two-component links, but we provide details for 7_5^2 here. Since the components K_1 and K_2 are of different knot types, we conclude that no symmetry in $\Sigma(7_5^2)$ can contain the permutation (12). Since $K_2 \approx K_2^*$, we cannot mirror L , i.e., the first entry of $\gamma \in \Sigma(7_5^2)$ cannot equal -1 . The linking number of L is nonzero, so we can rule out the symmetries $(1, -1, 1, e)$ and $(1, 1, -1, e)$ by Lemma 5.6. Last, L is purely invertible, meaning isotopic to $-L = -K_1 \cup -K_2$. Thus, $\Sigma(L)$ is the two element group $\Sigma_{2,1} = \{(1, 1, 1, e), (1, -1, -1, e)\}$. There are 8 cosets of this two element group in the 16 element group Γ_2 , so there are 8 isotopy classes of 7_5^2 links.

We now prove

Proposition 2.9. *The Whitten symmetry group $\Sigma(L)$ is the image of $\text{Sym}(L)$ under the map $\pi : \text{Sym}(L) = \text{MCG}(S^3, L) \rightarrow \text{MCG}(S^3) \times \text{MCG}(L) = \Gamma(L)$.*

Proof. Given a map $f : S^3 \rightarrow S^3 \in \text{MCG}(S^3, L)$, we see that if f is orientation-preserving on S^3 , then it is homotopic to the identity on S^3 since $\text{MCG}(S^3) = \mathbf{Z}_2$. This homotopy yields an ambient isotopy between L and $f(L)$, proving that $f|_L = \pi(f) \in \Sigma(L)$. If f is orientation-reversing on S^3 , it is homotopic to a standard reflection r . Composing the homotopy with r provides an ambient isotopy between L and $r(f(L))$, proving that $\pi(f) \in \Sigma(L)$. This shows $\pi(\text{Sym}(L)) \subset \Sigma(L)$.

Now suppose $g \in \Sigma(L)$. The isotopy from L to $g(L)$ generates an orientation-preserving (since it is homotopic to the identity) diffeomorphism $f : S^3 \rightarrow S^3$ which either fixes L or takes L to rL . In the first case, $f \in \text{Sym}(L)$. In the second, the map $rf \in \text{Sym}(L)$. \square

3. THE LINKING MATRIX

For each link L , our overall strategy will be to explicitly give isotopies for certain elements of the symmetry subgroup $\Sigma(L)$, generate the subgroup containing those elements, and then rule out the remainder of $\Gamma(L)$ using invariants. For three- and four-component links, a great deal of information about $\Sigma(L)$ can usually be obtained by considering the collection of pairwise linking numbers of the components of the link.

We recall a few definitions:

Definition 3.1. The *linking number* $\text{Lk}(L_1, L_2)$ of a pair of curves L_1 and L_2 is the intersection number of L_1 with any oriented surface spanning L_2 . Given a n -component link, with components L_1, L_2, \dots, L_n , we let the $n \times n$ *linking matrix* of L be the matrix $\text{Lk}(L)$ so that $\text{Lk}(L)_{ij} = \text{Lk}(L)_{ji} = \text{Lk}(L_i, L_j)$ and $\text{Lk}(L)_{ii} = 0$. We let $\text{Lk}(n)$ denote the set of $n \times n$ symmetric, integer-valued matrices with zeros on the diagonal.

The linking number can also be computed by counting the signed crossings of one knot over another. Among minimal crossing number diagrams of alternating links, the following three numbers are also useful link invariants:

Definition 3.2. The (overall) *linking number* $\ell k(L)$ of L is half the sum of the entries of the linking matrix. (This is half of the intercomponent signed crossings of L .) The *writhe* $\text{wr}(L)$ is the sum of all signed crossings of L . The *self-writhe* $s(L)$ is the sum of the intracomponent signed crossings of L . Clearly, $\text{wr} = 2\ell k + s$.

Murasugi and Thistlethwaite separately showed that writhe was an invariant of reduced alternating link diagrams [Mur87, Thi88]. Since linking number is an invariant for all links, self-writhe is also an invariant of reduced alternating diagrams. We will utilize this invariance to rule out certain symmetries of links, cf. Lemma 5.6.

The Whitten group Γ_n acts on the set of linking matrices $\text{Lk}(n)$. Further, for a given link L , the symmetry subgroup $\Sigma(L)$ must be a subgroup of the stabilizer of $\text{Lk}(L)$ under this action. This means that it is worthwhile for us to understand this action and make a classification of linking matrices according to their orbit types. We start by writing down the action:

Proposition 3.3. *The action of the Whitten group Γ_n on n -component links gives rise to the following Γ_n action on the set of $n \times n$ linking matrices $\text{Lk}(n)$:*

$$\text{Lk}(\gamma(L))_{ij} = \epsilon_0 \epsilon_i \epsilon_j \text{Lk}(L)_{p(i)p(j)}.$$

Proof. Equation (1) reminds us that

$$\gamma(L) = \epsilon_1 K_{p(1)}^{(*)} \cup \epsilon_2 K_{p(2)}^{(*)} \cup \dots \cup \epsilon_\mu K_{p(\mu)}^{(*)} = \bigcup_{i=1}^{\mu} \epsilon_i K_{p(i)}^{(*)}$$

This means that the i th component of $\gamma(L)$ is component $p(i)$ of L . Recall that linking number is reversed by changing the orientation of either curve or the ambient S^3 , which proves that we should multiply by $\epsilon_0\epsilon_i\epsilon_j$ as claimed. \square

Corollary 3.4. *If $\gamma = (\epsilon_0, \epsilon_1, \dots, \epsilon_3, p)$, let $\epsilon = \epsilon_0\epsilon_1\epsilon_2\epsilon_3$. The action of Γ_3 on $\text{Lk}(3)$ can be written*

$$\text{Lk}(\gamma(L))_{ij} = \epsilon_k \epsilon \text{Lk}(L)_{p(i)p(j)}.$$

where $\{i, j, k\} = \{1, 2, 3\}$ as sets.

In principle, this description of the action provides all the information one needs to compute orbits and stabilizers for any given matrix (for instance, by computer). However, that brute force approach doesn't yield much insight into the structure of the problem. We now develop enough theory to understand the situation without computer assistance in the case of three-component links.

3.1. Linking matrix for three-component links. We first observe that there is a bijection between $\text{Lk}(3)$ and \mathbf{Z}^3 given by

$$(2) \quad \begin{pmatrix} 0 & z_3 & z_2 \\ z_3 & 0 & z_1 \\ z_2 & z_1 & 0 \end{pmatrix} \leftrightarrow (z_1, z_2, z_3)$$

We would like to understand the action of Γ_3 on $\text{Lk}(3)$ by reducing it to the natural action of the simpler group $(\mathbf{Z}_2)^3 \rtimes S_3$ on \mathbf{Z}^3 . We start by building a homomorphism from Γ_3 onto $(\mathbf{Z}_2)^3 \rtimes S_3$.

Proposition 3.5. *Given an element $\gamma = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, p) \in \Gamma_3$, let $\epsilon = \epsilon_0\epsilon_1\epsilon_2\epsilon_3$. Then there is a homomorphism f from Γ_3 onto the group $(\mathbf{Z}_2)^3 \rtimes S_3$, given by*

$$f : (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, p) \mapsto (\epsilon_1\epsilon, \epsilon_2\epsilon, \epsilon_3\epsilon, p)$$

with kernel the 2-element subgroup $\{(1, 1, 1, 1, e), (1, -1, -1, -1, e)\}$ of Γ_3 . The preimage of an element $(\delta_1, \delta_2, \delta_3, p)$ is given by $(\delta_1\delta_2\delta_3, \delta_1, \delta_2, \delta_3, p)$ and $(\delta_1\delta_2\delta_3, -\delta_1, -\delta_2, -\delta_3, p)$.

Proof. We need to check that $f(\gamma * \gamma') = f(\gamma) * f(\gamma')$. Now $\gamma * \gamma' = (\epsilon_0\epsilon'_0, \epsilon_1\epsilon'_{p(1)}, \epsilon_2\epsilon'_{p(2)}, \dots, \epsilon_\mu\epsilon'_{p(\mu)}, qp)$. This means that

$$\begin{aligned} f(\gamma * \gamma') &= (\epsilon_1\epsilon'_{p(1)}\epsilon_0\epsilon'_0\epsilon_1\epsilon'_{p(1)}\epsilon_2\epsilon'_{p(2)}\epsilon_3\epsilon'_{p(3)}, \dots, qp) \\ &= (\epsilon_1\epsilon'_{p(1)}\epsilon\epsilon', \epsilon_2\epsilon'_{p(2)}\epsilon\epsilon', \epsilon_3\epsilon'_{p(3)}\epsilon\epsilon', qp), \end{aligned}$$

since $\epsilon'_{p(1)}\epsilon'_{p(2)}\epsilon'_{p(3)} = \epsilon'_1\epsilon'_2\epsilon'_3$ for any permutation p . But

$$\begin{aligned} f(\gamma) * f(\gamma') &= (\epsilon_1\epsilon, \dots, \epsilon_3\epsilon, p) * (\epsilon'_1\epsilon', \dots, \epsilon'_3\epsilon', q) \\ &= (\epsilon_1\epsilon\epsilon'_{p(1)}\epsilon', \dots, \epsilon_3\epsilon\epsilon'_{p(3)}\epsilon', qp). \end{aligned}$$

This proves that f is a homomorphism. Before computing its kernel, consider the image $(\delta_1, \delta_2, \delta_3, p) = f(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, p)$. Then the product $\delta_1\delta_2\delta_3 = \epsilon_0$ since $\delta_1\delta_2\delta_3 = \epsilon_1\epsilon_2\epsilon_3\epsilon = \epsilon_0^3\epsilon_1^4\epsilon_2^4\epsilon_3^4 = \epsilon_0$.

So suppose that $f(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, p) = (1, 1, 1, e)$. It is clear that $p = e$ and $\epsilon_0 = 1$. Since $\epsilon_1\epsilon = \epsilon_2\epsilon = \epsilon_3\epsilon = 1$, if $\epsilon_1 = 1$, then $\epsilon_2 = \epsilon_3 = 1$ and $\gamma = (1, 1, 1, 1, e)$. Likewise, if $\epsilon_1 = -1$, then $\epsilon_2 = \epsilon_3 = -1$ as well, and $\gamma = (1, -1, -1, -1, e)$.

Since Γ_3 is a group of order 96 and the kernel of f has order 2, the image of f has order 48. Since the target group $(\mathbf{Z}_2)^3 \rtimes S_3$ also has order 48, we conclude that f is surjective, as claimed. A direct computation verifies that the preimages above are correct. \square

We now need to show that the action of Γ_3 on $\text{Lk}(3)$ corresponds to the natural action of $(\mathbf{Z}_2)^3 \rtimes S_3$ on \mathbf{Z}^3 under this homomorphism.

Proposition 3.6. For any $\gamma \in \Gamma_3$, if $f : \Gamma_3 \rightarrow (\mathbf{Z}_2)^3 \rtimes S_3$ is the homomorphism of Proposition 3.5 and \leftrightarrow is the correspondence between $\text{Lk}(3)$ and \mathbf{Z}^3 of (2), we have

$$\gamma * \begin{pmatrix} 0 & z_3 & z_2 \\ z_3 & 0 & z_1 \\ z_2 & z_1 & 0 \end{pmatrix} \leftrightarrow f(\gamma) * (z_1, z_2, z_3).$$

Proof. Let $\gamma = (\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, p)$. By Corollary 3.4, the Γ_3 action (on the left-hand side) maps each entry $z_k = \text{Lk}(L)_{ij}$ in the linking matrix to $\epsilon\epsilon_k \text{Lk}(L)_{p(i)p(j)} = \epsilon\epsilon_k z_{p(k)}$, i.e.,

$$(3) \quad \gamma * \begin{pmatrix} 0 & z_3 & z_2 \\ z_3 & 0 & z_1 \\ z_2 & z_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \epsilon\epsilon_3 z_{p(3)} & \epsilon\epsilon_2 z_{p(2)} \\ \epsilon\epsilon_3 z_{p(3)} & 0 & \epsilon\epsilon_1 z_{p(1)} \\ \epsilon\epsilon_2 z_{p(2)} & \epsilon\epsilon_1 z_{p(1)} & 0 \end{pmatrix}.$$

On the right-hand side, by the definition of f , the natural action of $(\mathbf{Z}_2)^3 \rtimes S_3$ on \mathbf{Z}^3 is

$$f(\gamma) * (z_1, z_2, z_3) = (\epsilon_1 \epsilon z_{p(1)}, \epsilon_2 \epsilon z_{p(2)}, \epsilon_3 \epsilon z_{p(3)}).$$

This triple corresponds precisely to the new linking matrix (3) obtained from the Γ_3 action, so we have shown the two actions correspond. \square

We are now in a position to classify 3×3 linking matrices according to their orbit types, and compute their stabilizers in Γ_3 . The stabilizer of a linking matrix $A \in \text{Lk}(3)$ as a subgroup of $\Gamma(3)$ under the group action of Proposition 3.3 is the preimage under the homomorphism f of Proposition 3.5 of the stabilizer of the corresponding triple $(z_1, z_2, z_3) \in \mathbf{Z}^3$ under the natural action of $(\mathbf{Z}_2)^3 \rtimes S_3$ on \mathbf{Z}^3 . Since the kernel of f has order 2, stabilizers in Γ_3 are twice the size of the corresponding stabilizers in $(\mathbf{Z}_2)^3 \rtimes S_3$.

There are 10 orbit types of triples (z_1, z_2, z_3) under this action. To list the orbit types, we write a representative triple in terms of variables a, b , and c which are assumed to be integers with distinct nonzero magnitudes. To list the stabilizers, we either give the group explicitly as a subgroup of $(\mathbf{Z}_2)^3 \rtimes S_3$ or provide a list of generators in the form $\langle g_1, g_2, \dots, g_n \rangle$. One of these groups, $S(a, a, -a)$, is more complicated and is described below.

(z_1, z_2, z_3)	Stabilizer in $(\mathbf{Z}_2)^3 \rtimes S_3$	Stab. in Γ_3	Order
$(0, 0, 0)$	$(\mathbf{Z}_2)^3 \rtimes S_3$	Γ_3	96
$(a, 0, 0)$	$(\{+1\} \times \mathbf{Z}_2 \times \mathbf{Z}_2) \rtimes \{e, (23)\}$	$D_4 \times \mathbf{Z}_2$	16
$(a, -a, 0)$	$\langle (1, 1, -1, e), (-1, -1, 1, (12)) \rangle$	$(\mathbf{Z}_2)^3$	8
$(a, a, 0)$	$(\{+1\} \times \{+1\} \times \mathbf{Z}_2) \rtimes \{e, (12)\}$	$(\mathbf{Z}_2)^3$	8
$(a, b, 0)$	$(\{+1\} \times \{+1\} \times \mathbf{Z}_2) \rtimes \{e\}$	D_2	4
$(a, a, -a)$	$S(a, a, -a)$	$\mathbf{Z}_2 \times S_3$	12
(a, a, a)	$(\{+1\} \times \{+1\} \times \{+1\}) \rtimes S_3$	$\mathbf{Z}_2 \times S_3$	12
$(a, b, -b)$	$\langle (1, -1, -1, (23)) \rangle$	D_2	4
(a, b, b)	$(\{+1\} \times \{+1\} \times \{+1\}) \rtimes \{e, (23)\}$	D_2	4
(a, b, c)	$\{(1, 1, 1, e)\}$	D_1	2

TABLE 4. This table gives stabilizers for triples in \mathbf{Z}^3 in the group $(\mathbf{Z}_2)^3 \rtimes S_3$ under the natural action of this group on \mathbf{Z}^3 . This list of examples covers all the orbit types of this action. As we have shown above, the preimages of these stabilizers in Γ_3 are the stabilizers of the corresponding linking matrices in $\text{Lk}(3)$. For convenience, the order of these preimages are given in the right-hand column of the table.

The group $S(a, a, -a)$ is a 6 element group isomorphic to S_3 (or D_3) given by

$$S(a, a, -a) = \left\{ \begin{array}{lll} (1, 1, 1, e) & (1, 1, 1, (12)) & (1, -1, -1, (23)) \\ (-1, 1, -1, (13)) & (1, -1, -1, (123)) & (-1, 1, -1, (132)) \end{array} \right\}$$

Using the preimage formula of Proposition 3.5, it is easy to compute the stabilizer of a given linking matrix in Γ_3 directly from the table above; we simply conjugate by a permutation to bring the corresponding triple into one of the forms above and then apply the preimage formula.

We can now draw some amusing conclusions which might not be obvious otherwise, such as:

Lemma 3.7. *If L is a three-component link and any element of $\Sigma(L)$ reverses orientation on S^3 then at least one pair of components of L has linking number zero.*

Proof. The stabilizer of $\text{Lk}(L)$ includes an element of the form $(-1, \epsilon_1, \epsilon_2, \epsilon_3, p)$ if and only if some element $(\delta_1, \delta_2, \delta_3, p)$ in the stabilizer of the corresponding triple (z_1, z_2, z_3) has $\delta_1\delta_2\delta_3 = -1$ since we showed in the proof of Proposition 3.5 that ϵ_0 equaled $\delta_1\delta_2\delta_3$.

A negative δ_i will switch the sign of the linking number $z_{p(i)}$; to stabilize the triple (z_1, z_2, z_3) there must be an even number of sign changes unless some $z_i = 0$. Hence, if L has some mirror symmetry, i.e., one with $\epsilon_0 = -1$, then $\delta_1\delta_2\delta_3 = -1$ which produces an odd number of sign changes, so some linking number $z_i = 0$. \square

Example 3.8. We will see that the linking matrix for 7_1^3 has corresponding triple in the form $(a, a, -a)$. This means that the stabilizer of this linking matrix is a group of order 12 isomorphic to $\mathbf{Z}_2 \times S_3$ in Γ_3 conjugate to the preimage of the stabilizer $S(a, a, -a)$. Using the preimage formula of Proposition 3.5, we can explicitly compute

$$f^{-1}(S(a, a, -a)) = \left\{ \begin{array}{lll} (1, 1, 1, 1, e) & (1, 1, 1, 1, (12)) & (1, 1, -1, -1, (23)) \\ (1, -1, -1, -1, e) & (1, -1, -1, -1, (12)) & (1, -1, 1, 1, (23)) \\ (1, -1, 1, -1, (13)) & (1, 1, -1, -1, (123)) & (1, -1, 1, -1, (132)) \\ (1, 1, -1, 1, (13)) & (1, -1, 1, 1, (123)) & (1, 1, -1, 1, (132)) \end{array} \right\}.$$

Conjugating this subgroup by $(1, 1, 1, 1, (13))$, we obtain the stabilizer of $\text{Lk}(7_1^3)$:

$$\left\{ \begin{array}{lll} (1, 1, 1, 1, e) & (1, 1, 1, 1, (23)) & (1, -1, -1, 1, (12)) \\ (1, -1, -1, -1, e) & (1, -1, -1, -1, (23)) & (1, 1, 1, -1, (12)) \\ (1, -1, 1, -1, (13)) & (1, -1, -1, 1, (132)) & (1, -1, 1, -1, (123)) \\ (1, 1, -1, 1, (13)) & (1, 1, 1, -1, (132)) & (1, 1, -1, 1, (123)) \end{array} \right\}$$

We know that $\Sigma(7_1^3)$ is a subgroup of this stabilizer; actually, it equals the stabilizer, which we show in section 6.4.

3.2. Linking matrix for four-component links. For four-component links, we will need to develop a different theory. There are only three prime four-component links with 8 or fewer crossings, and fortunately they all possess a particular type of linking matrix. While the situation seems too complicated to make a full analysis of the S_4 action on the 6 nonzero elements of a general 4×4 linking matrix, it is relatively simple to come up with a theory which covers our cases. We first give a correspondence between certain 4×4 linking matrices and elements of \mathbf{Z}^4 :

$$(4) \quad \begin{pmatrix} 0 & z_1 & z_3 & 0 \\ z_1 & 0 & 0 & z_2 \\ z_3 & 0 & 0 & z_4 \\ 0 & z_2 & z_4 & 0 \end{pmatrix} \leftrightarrow (z_1, z_2, z_3, z_4)$$

Equivalently, we let

$$z_1 = A_{12}, \quad z_2 = A_{24}, \quad z_3 = A_{31}, \quad z_4 = A_{43}.$$

If we let Γ_4 act on the matrix A_{ij} as usual, matrices in this form are fixed by the subgroup with permutations in an 8 element subgroup of S_4 isomorphic to D_4 which we will call G_0 . As before, there is a standard homomorphism from this subgroup of Γ_4 to $(\mathbf{Z}_2)^4 \rtimes S_4$ given by

Proposition 3.9. *Let G_0 denote the subgroup $\{e, (14), (23), (14)(23), (12)(34), (13)(24), (1243), (1342)\}$ (isomorphic to D_4) of S_4 , and $f_0 : G_0 \rightarrow G_0$ denote the isomorphism given by*

$$\begin{array}{cccc} e \mapsto e & (14) \mapsto (12)(34) & (13)(24) \mapsto (14) & (1243) \mapsto (1243) \\ & (23) \mapsto (13)(24) & (12)(34) \mapsto (23) & (14)(23) \mapsto (14)(23) \\ & & & (1342) \mapsto (1342) \end{array}$$

There is a homomorphism f from the subgroup

$$G = \{(\epsilon_0, \epsilon_1, \dots, \epsilon_4, p) | p \in G_0\} < \Gamma_4$$

to $(\mathbf{Z}_2)^4 \rtimes S_4$ given by

$$(\epsilon_0, \epsilon_1, \dots, \epsilon_4, p) \mapsto (\epsilon_0\epsilon_1\epsilon_2, \epsilon_0\epsilon_2\epsilon_4, \epsilon_0\epsilon_1\epsilon_3, \epsilon_0\epsilon_3\epsilon_4, f_0(p)).$$

This homomorphism has a kernel of order 4 given by

$$\ker(f) = \{(1, 1, 1, 1, 1, e), (1, -1, -1, -1, -1, e), (-1, -1, 1, 1, -1, e), (-1, 1, -1, -1, 1, e)\}.$$

The image of f in $(\mathbf{Z}_2)^4 \rtimes S_4$ is $\{(\delta_1, \delta_2, \delta_3, \delta_4, f_0(p)) : \delta_4 = \delta_1\delta_2\delta_3, p \in G_0\}$. The preimage of such an element is the 4 element set $\{(\epsilon_1\epsilon_2\delta_1, \epsilon_1, \epsilon_2, \epsilon_2\delta_1\delta_3, \epsilon_1\delta_1\delta_2, p)\}$, where ϵ_1 and ϵ_2 are arbitrary.

Proof. A series of easy but lengthy direct computations. □

Proposition 3.10. *For any $\gamma \in G < \Gamma_4$ if f is the homomorphism of Proposition 3.9 and \leftrightarrow is the correspondence between $\text{Lk}(4)$ and \mathbf{Z}^4 given by Equation 4, we have*

$$(5) \quad \gamma * \begin{pmatrix} 0 & z_1 & z_3 & 0 \\ z_1 & 0 & 0 & z_2 \\ z_3 & 0 & 0 & z_4 \\ 0 & z_2 & z_4 & 0 \end{pmatrix} \leftrightarrow f(\gamma) * (z_1, z_2, z_3, z_4)$$

where γ acts by the action of Proposition 3.3 and $f(\gamma)$ acts by the natural action of $(\mathbf{Z}_2)^4 \rtimes S_4$ on \mathbf{Z}^4 .

Proof. It is clear from the correspondence between linking matrices and quadruples that the coefficients $(\delta_1, \delta_2, \delta_3, \delta_4)$ of $f(\gamma)$ are chosen so that the statement above holds. To complete the proof, we need only check that

$$z_{f_0(p(1))} = A_{p(1)p(2)}, \quad z_{f_0(p(2))} = A_{p(2)p(4)}, \quad z_{f_0(p(3))} = A_{p(3)p(1)}, \quad z_{f_0(p(4))} = A_{p(4)p(3)}$$

for all $p \in G_0$. This is a straightforward, if lengthy, computation. □

We now need to find stabilizers for a few carefully chosen linking matrix types.

Lemma 3.11. *The stabilizers in $f(G) < (\mathbf{Z}_2)^4 \rtimes S_4$ of (a, a, a, a) , $(a, a, -a, a)$, and $(a, -a, -a, a)$ are all 8 element groups isomorphic to D_4 . These produce 32-element stabilizers for the corresponding linking matrices in Γ_4 isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times D_4$. The individual stabilizers in $f(G)$ are in the form $(\delta_1, \dots, \delta_4, p)$ where the δ_i are determined uniquely by p . For (a, a, a, a) , the $\delta_i = 1$. In the other two cases, the pattern of signs is more intricate. We give the subgroups explicitly below in Table 5.*

As before, we can draw some conclusions about links from this theory which we might not have noticed otherwise. For example,

Corollary 3.12. *If L is a four-component link with linking matrix in the form of (4), then no element of $\Sigma(L)$ exchanges components 1 and 3 without also exchanging components 2 and 4.*

(z_1, z_2, z_3, z_4)	Stabilizer subgroup $S(z_1, z_2, z_3, z_4)$ of $f(G) < (\mathbf{Z}_2)^4 \rtimes S_4$		
(a, a, a, a)	$(\{+1\} \times \{+1\} \times \{+1\} \times \{+1\}) \rtimes G_0$		
$(a, a, -a, a)$	$(1, 1, 1, 1, e)$	$(1, 1, -1, -1, (12)(34))$	$(1, 1, -1, -1, (1243))$
	$(1, 1, 1, 1, (14))$	$(-1, 1, -1, 1, (13)(24))$	$(-1, 1, -1, 1, (1342))$
	$(1, -1, -1, 1, (23))$	$(1, -1, -1, 1, (14)(23))$	
$(a, -a, -a, a)$	$(1, 1, 1, 1, e)$	$(-1, -1, -1, -1, (12)(34))$	$(-1, -1, -1, -1, (1243))$
	$(1, 1, 1, 1, (14))$	$(-1, -1, -1, -1, (13)(24))$	$(-1, -1, -1, -1, (1342))$
	$(1, 1, 1, 1, (23))$	$(1, 1, 1, 1, (14)(23))$	

TABLE 5. Stabilizer subgroups for certain four-component links

4. THE SATELLITE LEMMA

We begin with two definitions.

Definition 4.1. A link L is *invertible* if reversing the orientation of all of its components produces a link isotopic to L , i.e., if $(1, -1, \dots, -1, p) \in \Sigma(L)$ for some permutation p . If the trivial permutation suffices, we call L *purely invertible*.

Definition 4.2. Let L be a link with μ components. If swapping the i th and j th components produces a link isotopic to L , i.e., if the element $\gamma = (1, \dots, 1, (ij)) \in \Sigma(L)$, then components (i, j) have a *pure exchange symmetry*.

We note that if components K_i and K_j are not of the same knot type, then it is impossible for them to have a pure exchange symmetry. We also note that if a link admits all pure exchange symmetries, the terms *invertible* and *purely invertible* are equivalent.

The most difficult part of our work below will be in ruling out pure exchange symmetries. So far, we have two (crude) tools; we can rule out pure exchange when the two components have different knot types or when the pure exchange does not preserve the linking matrix. In a few cases, these tools will not be enough and we will need the following:

Lemma 4.3. *Suppose that $L(K, i)$ is a satellite of L constructed by replacing component i with a knot or link K . Then L cannot have a pure exchange symmetry exchanging components i and j unless $L(K, i)$ and $L(K, j)$ are isotopic.*

Proof. Such a pure exchange would carry an oriented solid tube around L_i to a corresponding oriented solid tube around L_j . If we imagine K embedded in this tube, this generates an isotopy between $L(K, i)$ and $L(K, j)$. \square

The point of this lemma is that we can often distinguish $L(K, i)$ and $L(K, j)$ using classical invariants which are insensitive to the original labeling of the link. This seems like a general technique, and it would be interesting to explore this topic further.

5. TWO-COMPONENT LINKS

This section records the symmetry group $\Sigma(L)$ for all prime two-component links with eight or fewer crossings; there are 30 such links to consider. Our results are summarized in section 5.1, which names and lists the symmetry groups which appear (see Table 6). We count how frequently each group appears by crossing number in Table 7. The symmetry group for each link is listed in Tables 8 and 9, by group and by link, respectively. Proofs of these assertions appear in section 5.2

5.1. Symmetry names and results. The Whitten group $\Gamma_2 = \mathbf{Z}_2 \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \rtimes S_2)$ of all possible symmetries for two-component links is a nonabelian 16 element group isomorphic to $\mathbf{Z}_2 \times D_4$. The symmetry group $\Sigma(L)$ of a given link must form a subgroup of Γ_2 . There are 27 mutually nonconjugate subgroups of Γ_2 ; of these possibilities, only seven are realized as the symmetry subgroup of a prime link with 9 or fewer crossings (see Table 6). An eighth appears as the symmetry subgroup of a 10-crossing knot.

Question 5.1. *Do all 27 nonconjugate subgroups of Γ_2 appear as the symmetry group of some (possibly composite, split) link? Of some prime, non-split link?*

Hillmann [Hil86] provided examples for a few of these symmetry subgroups, but some of his examples were split links. Here are the groups we found among links with 8 or fewer crossings.

<i>Symmetry name</i>	<i>Notation</i>	<i>Subgroup of</i> $\Gamma_2 = \mathbf{Z}_2 \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \rtimes S_2)$	<i>Isomorphic to</i>
No symmetry	$\{e\}$	$\{(1, 1, 1, e)\}$	$\{e\}$
Purely Invertible	$\Sigma_{2,1}$	$\{1\} \times \Delta(\mathbf{Z}_2 \times \mathbf{Z}_2) \times \{e\}$	$\mathbf{Z}_2 \cong D_1$
Invertible with pure exchange	$\Sigma_{4,1}$	$\{1\} \times (\Delta(\mathbf{Z}_2 \times \mathbf{Z}_2) \rtimes S_2)$	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$
Individually invertible	$\Sigma_{4,2}$	$\{1\} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \{e\}$	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$
Even number of operations	$\Sigma_{4,3}$	$\{(\epsilon_0, \epsilon_1, \epsilon_2, e) : \epsilon_0\epsilon_1\epsilon_2 = 1\}$	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$
Full orientation-preserving	$\Sigma_{8,1}$	$\{1\} \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \rtimes S_2)$	D_4
Even ops & pure exchange	$\Sigma_{8,2}$	$\{(\epsilon_0, \epsilon_1, \epsilon_2, p) : \epsilon_0\epsilon_1\epsilon_2 = 1\}$	D_4
No exchanges	$\Sigma_{8,3}$	$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \{e\}$	$\mathbf{Z}_2 \times \mathbf{Z}_2 \times \mathbf{Z}_2$
Full symmetry	Γ_2	Γ_2	$\mathbf{Z}_2 \times D_4$

TABLE 6. Certain Whitten symmetry groups of two-component links (subgroups of Γ_2).

The first seven nontrivial groups in Table 6 are realized as the symmetry group of a link with nine or fewer crossings, while $\Sigma_{8,3}$ appears to be the symmetry group of a 10-crossing link. We know of no nontrivial links with full symmetry but speculate that they exist.

The next table records the frequency of each group.

Crossings	2-component links count	Link symmetry group							Full
		$\Sigma_{2,1}$	$\Sigma_{4,1}$	$\Sigma_{4,2}$	$\Sigma_{4,3}$	$\Sigma_{8,1}$	$\Sigma_{8,2}$	$\Sigma_{8,3}$	
0a	1								1
2a	1						1		
4a	1		1						
5a	1					1			
6a	3		2				1		
7a	6	1	2	2		1			
7n	2	1		1					
8a	14	3	7	3			1		
8n	2	1		1					
Total	31	6	12	7		2	3		1
First example		7_5^2	4_1^2	7_4^2	9_{61}^2	5_1^2	2_1^2	-	0_1^2

TABLE 7. Whitten symmetry groups, by crossing number and by alternation

Next, Table 8 lists the prime two-component links of eight or fewer crossings by symmetry group.

<i>Symmetry name</i>	<i>Notation</i>	<i>Prime links</i>
No symmetry	$\{e\}$	none
Invertible	$\Sigma_{2,1}$	$7_5^2, 7_7^2, 8_9^2, 8_{11}^2, 8_{14}^2, 8_{16}^2$
Invertible with pure exchange	$\Sigma_{4,1}$	$4_1^2, 6_1^2, 6_3^2, 7_1^2, 7_2^2, 8_1^2, 8_2^2, 8_3^2, 8_4^2, 8_5^2, 8_6^2, 8_7^2$
Individually invertible	$\Sigma_{4,2}$	$7_4^2, 7_6^2, 7_8^2, 8_{10}^2, 8_{12}^2, 8_{13}^2, 8_{15}^2$
Even number of operations	$\Sigma_{4,3}$	none
Full orientation-preserving	$\Sigma_{8,1}$	$5_1^2, 7_3^2$
Even operations with pure exchange	$\Sigma_{8,2}$	$2_1^2, 6_2^2, 8_8^2$
No exchanges	$\Sigma_{8,3}$	none
Full symmetry	Γ_2	0_1^2

TABLE 8. List of Whitten symmetry groups, and links possessing that symmetry

We conclude our tables of results by listing each two-component link and its corresponding Whitten symmetry group:

Link	Symmetry Group	Link	Symmetry Group	Link	Symmetry Group
2_1^2 (2a1)	$\Sigma_{8,2}$	7_5^2 (7a2)	$\Sigma_{2,1}$	8_7^2 (8a8)	$\Sigma_{4,1}$
4_1^2 (4a1)	$\Sigma_{4,1}$	7_6^2 (7a1)	$\Sigma_{4,2}$	8_8^2 (8a9)	$\Sigma_{8,2}$
5_1^2 (5a1)	$\Sigma_{8,1}$	7_7^2 (7n1)	$\Sigma_{2,1}$	8_9^2 (8a3)	$\Sigma_{2,1}$
6_1^2 (6a3)	$\Sigma_{4,1}$	7_8^2 (7n2)	$\Sigma_{4,2}$	8_{10}^2 (8a2)	$\Sigma_{4,2}$
6_2^2 (6a2)	$\Sigma_{8,2}$	8_1^2 (8a14)	$\Sigma_{4,1}$	8_{11}^2 (8a5)	$\Sigma_{2,1}$
6_3^2 (6a1)	$\Sigma_{4,1}$	8_2^2 (8a12)	$\Sigma_{4,1}$	8_{12}^2 (8a4)	$\Sigma_{4,2}$
7_1^2 (7a6)	$\Sigma_{4,1}$	8_3^2 (8a11)	$\Sigma_{4,1}$	8_{13}^2 (8a1)	$\Sigma_{4,2}$
7_2^2 (7a5)	$\Sigma_{4,1}$	8_4^2 (8a13)	$\Sigma_{4,1}$	8_{14}^2 (8a7)	$\Sigma_{2,1}$
7_3^2 (7a4)	$\Sigma_{8,1}$	8_5^2 (8a10)	$\Sigma_{4,1}$	8_{15}^2 (8n2)	$\Sigma_{4,2}$
7_4^2 (7a3)	$\Sigma_{4,2}$	8_6^2 (8a6)	$\Sigma_{4,1}$	8_{16}^2 (8n1)	$\Sigma_{2,1}$

TABLE 9. The Whitten symmetry group for each two-component link

Theorem 5.2. *The symmetry groups for all prime two-component links up to 8 crossings are as listed in Table 9.*

The proof of this theorem is divided into five cases, based on the five symmetry groups that appear in Table 9; these proofs are found in section 5.2. Many of our arguments generalize to various families of links. As this paper focuses on these first examples, we ask for the reader's understanding when we eschew the most general argument in favor of a simpler, more expedient one.

5.2. Proofs for two-component links. Below, we attempt to provide a general framework for determining symmetry groups for two-component links. Since $\Sigma(L)$ is a subgroup of Γ_2 , the order of the symmetry group must divide $|\Gamma_2| = 16$. Our strategy begins by exhibiting certain symmetries via explicit isotopies. With these in hand, we next use various techniques to rule out some symmetries until we can finally determine the symmetry group $\Sigma(L)$. These techniques generally involve using some link invariant to show $L^\gamma \approx L$. Among link invariants, the linking number and self-writhe (for alternating links) are easily applied since they count signed crossings; we also use polynomials and other methods.

We focus on the 30 prime links with eight or fewer crossings. Our first results indicate which of these 30 links have either a *pure invertibility* or a *pure exchange* symmetries, which we prove explicitly by exhibiting isotopies. Recall that a link is purely invertible if reversing all components' orientations produces an isotopic link; a link has pure exchange symmetry if swapping its two components is an isotopy.

Lemma 5.3. *Via the isotopies exhibited in Figures 6, 7, and 20-26 in Appendices B.1 and C.1, the following 17 links have pure exchange symmetry:*

$$(6) \quad 2_1^2, 4_1^2, 5_1^2, 6_1^2, 6_2^2, 6_3^2, 7_1^2, 7_2^2, 7_3^2, 8_1^2, 8_2^2, 8_3^2, 8_4^2, 8_5^2, 8_6^2, 8_7^2, 8_8^2$$

i.e., $(1, 1, 1, (12))$ belongs to the symmetry group of each of these links.

As we determine symmetry groups, we will establish that the remaining 13 links in consideration do not have pure exchange symmetry.

Cerf [Cer98] states that all prime, alternating two-component links of 8 or fewer crossings are invertible, though this may involve exchanging components. Via the isotopies exhibited in Figures 8 and 27-34 of Appendices B.2 and C.2, we extend Cerf's result to non-alternating links, and we show that the invertibility is pure (i.e., without exchanging components). To obtain invertibility for 7_8^2 , combine the results of Figures 9 and 10, which show that each of its components can be individually inverted.

Lemma 5.4. *All 30 prime two-component links with eight or fewer crossings are purely invertible.*

We note that the pure exchange and pure invertibility symmetries, corresponding to Whitten elements $(1, 1, 1, (12))$ and $(1, -1, -1, e)$, respectively, generate the subgroup $\Sigma_{4,1}$ of Γ_2 . This implies our first result about link symmetry groups.

Lemma 5.5. *Any two-component link, such as those listed in (6), that has both pure exchange symmetry and (pure) invertibility, must have $\Sigma_{4,1}$ as a subgroup of its symmetry group $\Sigma(L)$.*

By examining signed crossings of a link, we calculate its linking number and self-writhe; if one of these is nonzero, we may rule out some symmetries.

- Lemma 5.6.**
- (1) *If the linking number $\text{Lk}(L) \neq 0$, then $\Sigma(L) < \Sigma_{8,2}$.*
 - (2) *For L alternating, if the self-writhe $s(L) \neq 0$, then $\Sigma(L) < \Sigma_{8,1}$.*
 - (3) *For L alternating, if $\text{Lk}(L) \neq 0$ and $s(L) \neq 0$, then $\Sigma(L) < \Sigma_{4,1}$.*

Proof. Consider the effect of each symmetry operation upon linking number (see Proposition 3.3): mirroring a link or inverting one of its components will swap the sign of the linking number, while exchanging its components fixes the linking number. As for the self-writhe of a link, it is fixed by inverting any component or exchanging the two components; however, mirroring a link swaps the sign of $s(L)$.

Thus, the elements of Γ_2 that will swap the sign of a linking number are precisely those of the form $\gamma = (\epsilon_0, \epsilon_1, \epsilon_2, p)$ with $\epsilon := \epsilon_0\epsilon_1\epsilon_2 = -1$. If the linking number $\text{Lk}(L)$ is nonzero, these cannot possibly produce a link L^γ isotopic to the original link L , so these eight elements are not part of $\Sigma(L)$. The remaining eight symmetry elements form $\Sigma_{8,2} = \{(\epsilon_0, \epsilon_1, \epsilon_2, p) : \epsilon_0\epsilon_1\epsilon_2 = 1\}$, which proves the first assertion.

Self-writhe is an invariant of reduced diagrams of alternating links, and any symmetry operation which mirrors the link will swap the sign of $s(L)$. If the self-writhe $s(L)$ is nonzero, then no element which mirrors, i.e., $(-1, \epsilon_1, \epsilon_2, p)$, can lie in $\Sigma(L)$. The remaining elements form $\Sigma_{8,1} = \{1\} \times (\mathbf{Z}_2 \times \mathbf{Z}_2 \times S_2)$.

The last assertion follows as an immediate corollary of the first two. If both hypotheses are satisfied, then $\Sigma(L) \subset \Sigma_{8,1} \cap \Sigma_{8,2} = \Sigma_{4,1}$. \square

Lemma 5.7. *Let L be a two-component link.*

- (1) *If L is invertible, then $\Sigma_{2,1} < \Sigma(L)$.*
- (2) *If the components of L are different knot types, then $\Sigma(L) < \Sigma_{8,3}$.*
- (3) *If both hypotheses above are true, and*
 - (a) *if $\text{lk}(L) \neq 0$, then $\Sigma(L)$ is either $\Sigma_{2,1}$ or $\Sigma_{4,3}$.*
 - (b) *if L is alternating and $s(L) \neq 0$, then $\Sigma(L)$ is either $\Sigma_{2,1}$ or $\Sigma_{4,2}$.*

Proof. The first assertion is immediate, since invertible symmetry generates $\Sigma_{2,1}$.

If the components of L have different knot types, then no exchange symmetries are permissible; the permutation $p = (12)$ never appears in $\Sigma(L)$. Hence the symmetry group $\Sigma(L)$ is contained in the 'No exchanges' group $\Sigma_{8,3}$.

Combining these two results with the previous lemma proves the third assertion. If the linking number is nonzero and the components of L have different knot types, then $\Sigma_{2,1} < \Sigma(L) < \Sigma_{8,2} \cap \Sigma_{8,3} = \Sigma_{4,3}$. If L is also invertible, then $\Sigma_{2,1} < \Sigma(L) < \Sigma_{4,3}$. This implies that the order of $\Sigma(L)$ equals 2 or 4, so it is either $\Sigma_{2,1}$ or $\Sigma_{4,3}$.

If instead self-writhe is nonzero and the first two hypotheses hold, then $\Sigma_{2,1} < \Sigma(L) < \Sigma_{8,1} \cap \Sigma_{8,3} = \Sigma_{4,2}$. This implies that the order of $\Sigma(L)$ equals 2 or 4, so it is either $\Sigma_{2,1}$ or $\Sigma_{4,2}$. \square

With these five lemmas in hand, we are now prepared to begin proving Theorem 5.2, which we treat by each symmetry group.

5.2.1. Links with symmetry group $\Sigma_{2,1}$.

Claim 5.8. Links 7_5^2 , 7_7^2 , 8_9^2 , 8_{11}^2 , 8_{14}^2 , and 8_{16}^2 have symmetry group $\Sigma_{2,1}$.

Proof. All of these links are purely invertible, so $\Sigma_{2,1} \subset \Sigma(L)$. Also, all of them have components of different knot types and nonzero linking numbers; thus by Lemma 5.7, their symmetry groups are either $\Sigma_{2,1}$ or $\Sigma_{4,3}$.

Three of the alternating links (7_5^2 , 8_{11}^2 , 8_{14}^2) have nonzero self-writhe, so we apply Lemma 5.7 again. We conclude that they have only the invertible symmetry, and $\Sigma_{2,1}$ is their symmetry group.

For the remaining three links in this case (7_7^2 , 8_{11}^2 , 8_{16}^2), consider the action of the Whitten element $\gamma = (-1, -1, 1, e)$. We consider the Jones polynomials of L and L^γ . They are unequal, as demonstrated below, which implies L^γ is not isotopic to L . Thus $\Sigma(L) \neq \Sigma_{4,3}$, so it must be $\Sigma_{2,1}$.

$$\begin{aligned} \text{Jones}(7_7^2) &= z^{-15/2} - z^{-13/2} - z^{-9/2} - z^{-5/2} \\ \text{Jones}((7_7^2)^\gamma) &= -z^{-7/2} - z^{-3/2} - \sqrt{z} + z^{3/2} \\ \\ \text{Jones}(8_{11}^2) &= -z^{9/2} + 3z^{7/2} - \frac{1}{z^{7/2}} - 4z^{5/2} + \frac{1}{z^{5/2}} + 5z^{3/2} - \frac{4}{z^{3/2}} - 5\sqrt{z} + \frac{4}{\sqrt{z}} \\ \text{Jones}((8_{11}^2)^\gamma) &= -\frac{4}{z^{9/2}} + \frac{1}{z^{7/2}} - \frac{1}{z^{5/2}} - \frac{1}{z^{21/2}} + \frac{3}{z^{19/2}} - \frac{4}{z^{17/2}} + \frac{5}{z^{15/2}} - \frac{5}{z^{13/2}} + \frac{4}{z^{11/2}} \\ \\ \text{Jones}(8_{16}^2) &= \frac{2}{z^{9/2}} - \frac{2}{z^{7/2}} + \frac{2}{z^{5/2}} - \frac{2}{z^{3/2}} - \frac{2}{z^{11/2}} - \sqrt{z} + \frac{1}{\sqrt{z}} \\ \text{Jones}((8_{16}^2)^\gamma) &= -\frac{2}{z^{9/2}} + \frac{2}{z^{7/2}} - \frac{2}{z^{5/2}} + \frac{2}{z^{3/2}} - \frac{1}{z^{13/2}} + \frac{1}{z^{11/2}} - \frac{2}{\sqrt{z}} \quad \square \end{aligned}$$

5.2.2. Links with symmetry group $\Sigma_{4,1}$.

Claim 5.9. Links 4_1^2 , 6_1^2 , 8_1^2 , 8_2^2 , and 8_4^2 have symmetry group $\Sigma_{4,1}$.

Proof. All five of these links appear in our list (6) of pure exchange symmetry links; also, they all are purely invertible and have nonzero linking numbers. Lemmas 5.3 and 5.6 imply that $\Sigma_{4,1} < \Sigma(L) < \Sigma_{8,2}$.

For each link, the Conway polynomials differ for L and L^γ , where $\gamma = (-1, -1, 1, e) \in \Sigma_{8,2}$. Thus each link cannot have $\Sigma_{8,2}$ as its symmetry group and must therefore have $\Sigma(L) = \Sigma_{4,1}$. We display the Conway polynomials in Table 10 below. \square

Link L	Conway(L)	Conway(L^γ)
4_1^2	$2z$	$z^3 + 2z$
6_1^2	$-z^5 - 4z^3 - 3z$	$-3z$
8_1^2	$-z^7 - 6z^5 - 10z^3 - 4z$	$-4z$
8_2^2	$2z^5 + 7z^3 + 4z$	$3z^3 + 4z$
8_4^2	$4z^3 + 4z$	$2z^5 + 6z^3 + 4z$

TABLE 10. Conway polynomials for Claim 5.9

Claim 5.10. Links 6_3^2 , 7_1^2 , 7_2^2 , 8_3^2 , 8_5^2 , 8_6^2 , and 8_7^2 have symmetry group $\Sigma_{4,1}$.

Proof. These links have both pure exchange and pure invertibility symmetries; they also have nonzero self-writhe and linking numbers. By Lemmas 5.3 and 5.6, their symmetry group must be $\Sigma_{4,1}$. \square

5.2.3. Links with symmetry group $\Sigma_{4,2}$.

Claim 5.11. Links 7_4^2 and 8_{12}^2 have symmetry group $\Sigma_{4,2}$.

Proof. These links are purely invertible, comprised of different knot types, and have self-writhe $s(L) \neq 0$; thus, Lemma 5.6 implies $\Sigma_{2,1} < \Sigma(L) < \Sigma_{4,2}$.

Figures 13 and 17 exhibit an isotopies which shows that $(-1, -1, 1, e) \in \Sigma(L)$ for each link, which means $\Sigma_{4,2}$ is their symmetry group. \square

Claim 5.12. Links 7_8^2 , 8_{10}^2 , and 8_{15}^2 have symmetry group $\Sigma_{4,2}$.

Proof. These links are purely invertible and comprised of different knot types; thus, Lemma 5.6 implies $\Sigma_{2,1} < \Sigma(L) < \Sigma_{8,3}$.

First, we use the Jones polynomial to rule out mirror symmetry, i.e., the element $\gamma = (-1, 1, 1, e) \in \Sigma_{8,3}$ does not lie in $\Sigma(L)$. That means order of the subgroup $\Sigma(L)$ is between 2 and 7; hence it is either a 2 or 4 element subgroup. Here are the Jones polynomials:

$$\begin{aligned} \text{Jones}(7_8^2) &= -\frac{1}{z^{9/2}} + \frac{1}{z^{7/2}} - \frac{2}{z^{5/2}} + \frac{1}{z^{3/2}} + \frac{1}{z^{11/2}} - \frac{2}{\sqrt{z}} \\ \text{Jones}((7_8^2)^\gamma) &= -z^{9/2} + z^{7/2} - 2z^{5/2} + z^{3/2} + z^{11/2} - 2\sqrt{z} \\ \\ \text{Jones}(8_{10}^2) &= -z^{9/2} + 3z^{7/2} - \frac{1}{z^{7/2}} - 5z^{5/2} + \frac{2}{z^{5/2}} + 5z^{3/2} - \frac{4}{z^{3/2}} - 6\sqrt{z} + \frac{5}{\sqrt{z}} \\ \text{Jones}((8_{10}^2)^\gamma) &= -\frac{1}{z^{9/2}} - z^{7/2} + \frac{3}{z^{7/2}} + 2z^{5/2} - \frac{5}{z^{5/2}} - 4z^{3/2} + \frac{5}{z^{3/2}} + 5\sqrt{z} - \frac{6}{\sqrt{z}} \\ \\ \text{Jones}(8_{15}^2) &= -\frac{1}{z^{7/2}} - z^{5/2} + \frac{1}{z^{5/2}} + z^{3/2} - \frac{1}{z^{3/2}} - 2\sqrt{z} + \frac{1}{\sqrt{z}} \\ \text{Jones}((8_{15}^2)^\gamma) &= -z^{7/2} + z^{5/2} - \frac{1}{z^{5/2}} - z^{3/2} + \frac{1}{z^{3/2}} + \sqrt{z} - \frac{2}{\sqrt{z}} \end{aligned}$$

Next, for each link we depict an isotopy which reverses the orientation of just one component, i.e., we show either $\gamma = (1, -1, 1, e)$ or $\gamma = (1, 1, -1, e)$ lies in $\Sigma(L)$. This means $\Sigma_{4,2}$ is the symmetry group for these three links.

Figures 9, 16, and 19 show these isotopies for 7_8^2 , 8_{10}^2 , and 8_{15}^2 , respectively. \square

Claim 5.13. Links 7_6^2 and 8_{13}^2 have symmetry group $\Sigma_{4,2}$.

Proof. These links are purely invertible and have nonzero self-writhe; thus Lemma 5.6 implies $\Sigma_{2,1} < \Sigma(L) < \Sigma_{8,1}$.

We take the satellites, L_1 , L_2 of the first and second component of L , for $L \in \{7_6^2, 8_{13}^2\}$, and compute the Jones polynomial for each.

$$\text{Jones}((8_{13}^2)_1) = 1 - \frac{1}{z^6} + \frac{2}{z^5} - \frac{3}{z^4} + \frac{4}{z^3} - \frac{2}{z^2} + \frac{1}{z} - 2z + 4z^2 - 2z^3 + 3z^4 - z^5$$

$$\text{Jones}((8_{13}^2)_2) = 1 + \frac{1}{z^{11}} - \frac{2}{z^{10}} + \frac{2}{z^8} - \frac{2}{z^7} + \frac{1}{z^6} + \frac{1}{z^5} - \frac{2}{z^4} + \frac{2}{z^3} - \frac{1}{z^2} + \frac{1}{z} - 2z + 4z^2 - z^3 + 2z^5 - z^6$$

$$\text{Jones}((7_6^2)_1) = 2 + 1/z^{10} - 2/z^9 + 1/z^8 - 1/z^6 + 2/z^5 - 1/z^4 + 2/z^3 - z + z^2 + z^3 - z^4$$

$$\text{Jones}((7_6^2)_2) = 3 + 1/z^7 - 2/z^6 + 2/z^5 - 2/z^4 + 2/z^3 + 1/z^2 - 2z + 2z^2 - z^3$$

Since the Jones polynomials the two different satellites are not equal for either link L , we have that L is not isotopic to L_γ for $\gamma = (1, 1, 1, (12))$ by Lemma 4.3. Thus, $(1, 1, 1, (12)) \notin \Sigma(L)$.

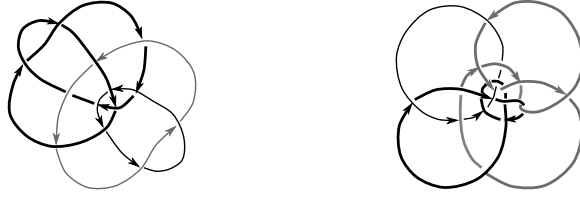


FIGURE 1. Gray knot is satellite for the first and second component of 7_6^2 , respectively

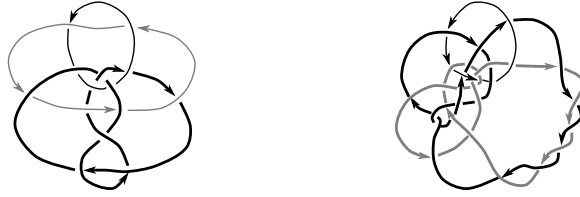


FIGURE 2. Gray knot is satellite for the first and second component of 8_{13}^2 , respectively

Figures 14 and 18 exhibit isotopies that show $(1, -1, 1, e) \in \Sigma(L)$ for both of these links. Therefore, we conclude $\Sigma(L) = \Sigma_{4,2}$ for these two links. \square

5.2.4. Links with symmetry group $\Sigma_{8,1}$.

Claim 5.14. Links 5_1^2 and 7_3^2 have symmetry group $\Sigma_{8,1}$.

Proof. These two links have the pure exchange and pure invertibility symmetries, and their self-writhe are nonzero; thus Lemma 5.6 implies $\Sigma_{4,1} < \Sigma(L) < \Sigma_{8,1}$. We show, in figures 10 and 9, that each symmetry group includes either $(1, 1, -1, e)$ or $(1, -1, 1, e)$, neither of which is an element of $\Sigma_{4,1}$. Therefore, we conclude $\Sigma(L) = \Sigma_{8,1}$ for these two links. \square

5.2.5. Links with symmetry group $\Sigma_{8,2}$.

Claim 5.15. Links 2_1^2 , 6_2^2 , and 8_8^2 have symmetry group $\Sigma_{8,2}$.

Proof. These three links have the pure exchange and pure invertibility symmetries, and their linking numbers are nonzero; thus Lemma 5.6 implies $\Sigma_{4,1} < \Sigma(L) < \Sigma_{8,2}$.

Figures 11, 12, and 15 display the isotopies which show $(-1, 1, -1, e)$ lies in the symmetry group for each of these three links. Since this element is not in $\Sigma_{4,1}$, we may conclude all three links have symmetry group $\Sigma(L) = \Sigma_{8,2}$. \square

6. THREE-COMPONENT LINKS

There are 14 three-component links with 8 or fewer crossings. In this section, we determine the symmetry group for each one; Table 11 summarizes the results. We obtain 11 different symmetry subgroups inside Γ_3 , which represent 7 different conjugacy classes of subgroups (out of the 131 possible).

For each link, our first task is to calculate the linking matrix. Then, we utilize Table 4 to determine the stabilizer of this matrix within Γ_3 ; we know that the symmetry group $\Sigma(L)$ must be a subgroup of this stabilizer. From there, we proceed by ruling out other elements using polynomial invariants and by exhibiting isotopies to show that certain symmetries do lie in $\Sigma(L)$ until we can discern the symmetry group.

Here are the results, listed in terms of generators for each group. We use the following notation for common group elements:

- PI, for pure invertibility, i.e., the element $(1, -1, -1, -1, e)$
- PE, for having all pure exchanges, i.e., all elements $(1, 1, 1, 1, p)$ where $p \in S_3$

Link(s)	$ \Sigma(L) $	Σ isomorphic to	Generators
$8_1^3(8a18), 8_8^3(8n4)$	4	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$	PI, $(1, 1, -1, -1, (23))$
$8_2^3(8a17), 8_7^3(8n3)$	4	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$	PI, $(1, 1, 1, 1, (23))$
$8_{10}^3(8n6)$	4	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$	PI, $(1, 1, 1, 1, (12))$
$8_4^3(8a20)$	4	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$	PI, $(-1, 1, 1, 1, (12))$
$8_5^3(8a16)$	4	$\mathbf{Z}_2 \times \mathbf{Z}_2 \cong D_2$	$(1, 1, -1, -1, e), (1, -1, -1, -1, (23))$
$8_6^3(8a19)$	8	$(\mathbf{Z}_2)^3$	PI, $(-1, 1, -1, -1, e), (1, 1, -1, -1, (23))$
$8_9^3(8n5)$	8	$(\mathbf{Z}_2)^3$	PI, $(1, 1, 1, 1, (23)), (1, 1, -1, -1, e)$
$6_1^3(6a5), 8_3^3(8a15)$	12	D_6	PI, PE
$6_3^3(6n1)$	12	D_6	$(1, 1, 1, 1, (12)), (1, -1, 1, 1, (123))$
$7_1^3(7a7)$	12	D_6	$(1, 1, 1, 1, (23)), (1, 1, -1, 1, (123))$
$6_2^3(6a4)$	48	$\mathbf{Z}_2 \times ((\mathbf{Z}_2)^2 \times S_3)$	$(-1, 1, 1, 1, e) \quad (1, 1, 1, 1, (123))$ $(1, -1, 1, 1, (12))$

TABLE 11. Three-component link symmetry groups, by crossing number. The first three groups are all conjugate to each other; also, all 12 element groups above are conjugate to each other.

We note that all but two of these links are purely invertible, even though PI might not be part of a minimal set of generators. Neither the Borromean rings (6_2^3) or the link 8_5^3 are purely invertible. Both of these links are, however, invertible. The Borromean rings can be inverted using any odd permutation p , i.e., they admit the symmetry $\gamma = (1, -1, -1, -1, p)$; the link 8_5^3 is invertible using $p = (23)$.

Claim 6.1. *The subgroup $\Sigma(6_1^3) < \Gamma_3$ is the 12 element group isomorphic to D_6 generated by pure exchanges and pure invertibility.*

Proof. The linking matrix for 6_1^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & -1 & -1 \\ 2 & -1 & 0 & -1 \\ 3 & -1 & -1 & 0 \end{array}$$

which is in the standard form (a, a, a) . We know that $\Sigma(6_1^3)$ is a subgroup of the stabilizer of this matrix under the action of Γ_3 on linking matrices. Consulting Table 4, we see that this stabilizer is the group in the claim. We must now show that all these elements are in the group. Figures 7 and 36 show that

$$(1, 1, 1, 1, (123)), (1, 1, 1, 1, (23)) \in \Sigma(6_1^3)$$

Since any 3-cycle and 2-cycle generate S_3 , we have the rest of the pure exchanges as well. Figure 35 shows that this link is purely invertible, completing the proof. \square

Claim 6.2. *The subgroup $\Sigma(6_2^3) < \Gamma_3$ is the 48 element group where $(\epsilon_0, \epsilon_1, \epsilon_2, \epsilon_3, p)$ is in the group if either*

- (a) $\epsilon_1\epsilon_2\epsilon_3 = 1$ and p is an even permutation, or
- (b) $\epsilon_1\epsilon_2\epsilon_3 = -1$ and p is odd.

Proof. Figures 10, 37, and 38 tell us that $\Sigma(6_2^3)$ contains the elements

$$(1, -1, 1, -1, (132)), (-1, -1, 1, 1, (13)), (-1, 1, 1, 1, e),$$

which clearly obey the rules in the claim. In fact, they generate a group of 48 such elements. Since the order of $\Sigma(6_1^3)$ must divide $|\Gamma_3| = 96$, it is either these 48 elements or it is all of Γ_3 . But in [Mon75], Montesinos proves that 6_2^3 is not purely invertible. Thus, $(1, -1, -1, -1, e)$ cannot be in $\Sigma(6_2^3)$, which completes the proof. \square

Claim 6.3. *The subgroup $\Sigma(6_3^3) < \Gamma_3$ is the 12 element group isomorphic to D_6 given by $f^{-1}(S(a, a, -a))$.*

Proof. Figures 39 and 40 imply that $\Sigma(6_3^3)$ contains

$$\{(1, 1, -1, 1, (132)), (1, -1, -1, -1, (12))\}.$$

These three elements generate the 12 element group of the claim. Now the linking matrix for 6_3^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & -1 \\ 3 & -1 & -1 & 0 \end{array}$$

which is in the standard form $(a, a, -a)$. Consulting Table 4, we see that this means $|\Sigma(6_3^3)|$ divides 12, the order of the stabilizer. Since we already have 12 elements in the symmetry group, it must equal the stabilizer, which completes the proof. \square

Claim 6.4. *The subgroup $\Sigma(7_1^3) < \Gamma_3$ is the 12 element subgroup isomorphic to D_6 that is conjugate to $f^{-1}(S(a, a, -a))$ by $(1, 1, 1, 1, (13))$.*

Proof. The linking matrix for 7_1^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & -1 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & -1 & 1 & 0 \end{array}$$

which corresponds to the triple $(1, -1, -1)$ and has the same orbit type as $(a, a, -a)$. Thus, the stabilizer of this linking matrix is a 12 element group conjugate to the stabilizer $f^{-1}(S(a, a, -a))$ by $(1, 1, 1, 1, (13))$. Figures 41 and 42 show that

$$(1, 1, -1, 1, (123)), (1, 1, 1, 1, (23)) \in \Sigma(7_1^3).$$

These elements generate a 12 element group, so this stabilizer is the entire symmetry group of 7_1^3 , as claimed. We note that this stabilizer was worked out explicitly as Example 3.8. \square

Claim 6.5. *The subgroup $\Sigma(8_1^3) < \Gamma_3$ is the 4 element group isomorphic to D_2 generated by pure invertibility and $(1, 1, -1, -1, (23))$.*

Proof. The linking matrix for 8_1^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & -1 & 1 \\ 2 & -1 & 0 & 2 \\ 3 & 1 & 2 & 0 \end{array}$$

which is in the standard form $(a, b, -b)$. Consulting Table 4, we see that the stabilizer of this linking matrix in Γ_3 has order 4. But Figures 8 and 43 show stabilizer elements

$$(1, -1, -1, -1, e), (1, -1, 1, 1, (23)) \in \Sigma(8_1^3),$$

which means that we also have $(1, 1, -1, -1, (23)) \in \Sigma(8_1^3)$. Therefore these three elements, plus the identity, must form the symmetry group $\Sigma(8_1^3)$. \square

Claim 6.6. *The subgroup $\Sigma(8_2^3) < \Gamma_3$ is the 4 element group isomorphic to D_2 generated by the pure exchange $(1, 1, 1, 1, (23))$ and pure invertibility.*

Proof. The linking matrix for 8_2^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & -1 & -1 \\ 2 & -1 & 0 & -2 \\ 3 & -1 & -2 & 0 \end{array}$$

which is in the standard form (a, b, b) . Consulting Table 4, we see that the stabilizer of $\text{Lk}(8_2^3)$ has order 4. Figures 9 and 44 show

$$(1, -1, -1, -1, (23)), (1, 1, 1, 1, (23)) \in \Sigma(8_2^3)$$

and these generate the 4 element subgroup of the claim, which equals the stabilizer. \square

Claim 6.7. *The subgroup $\Sigma(8_3^3) < \Gamma_3$ is the 12 element group isomorphic to D_6 generated by pure exchanges and pure invertibility.*

Proof. The linking matrix for 8_3^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & -1 & -1 \\ 2 & -1 & 0 & -1 \\ 3 & -1 & -1 & 0 \end{array}$$

which is in the form (a, a, a) . Consulting Table 4, we see that the stabilizer of $\text{Lk}(8_3^3)$ has order 12, and hence $|\Sigma(8_3^3)|$ divides 12. Now Figures 46 and 47 show that

$$(1, 1, 1, 1, (12)), (1, 1, 1, 1, (13)) \in \Sigma(8_3^3).$$

Since the cycles (12) and (13) generate all of S_3 , we know that all 6 of the pure exchanges are in $\Sigma(8_3^3)$. Figure 45 shows that 8_3^3 is invertible as well, completing the proof.

Finally, we note that we have encountered this symmetry group before, as $\Sigma(6_1^3) = \Sigma(8_3^3)$. \square

Claim 6.8. *The subgroup $\Sigma(8_4^3) < \Gamma(3)$ is the 4 element group isomorphic to D_2 generated by pure invertibility and $(-1, 1, 1, 1, (12))$.*

Proof. The linking matrix for 8_4^3 is

	1	2	3
1	0	0	2
2	0	0	-2
3	2	-2	0

which is in the standard form $(a, -a, 0)$. Consulting Table 4, we see that the stabilizer of $\text{Lk}(8_4^3)$ is the 8 element group $f^{-1}(S(a, -a, 0))$, which is generated by $(-1, 1, 1, -1, e)$, $(-1, 1, 1, 1, (12))$, and pure invertibility. Hence, $|\Sigma(8_4^3)|$ divides 8. Figures 8 and 48 show that the latter two of these generators, namely pure invertibility and $(-1, 1, 1, 1, (12))$, are in $\Sigma(8_4^3)$.

To finish the proof, we now show that the third stabilizer generator $(-1, 1, 1, -1, e)$ does not lie in the symmetry group. Applying it to 8_4^3 , we get a link with HOMFLYPT polynomial

$$a^4 + \frac{1}{a^4} - 2a^2z^2 + \frac{a^2}{z^2} - \frac{2z^2}{a^2} + \frac{1}{a^2z^2} + z^4 - \frac{2}{z^2} - 2.$$

However, the base 8_4^3 has HOMFLYPT polynomial

$$a^2z^4 + \frac{z^4}{a^2} + 3a^2z^2 + \frac{3z^2}{a^2} + \frac{a^2}{z^2} + \frac{1}{a^2z^2} + 4a^2 + \frac{4}{a^2} - z^6 - 5z^4 - 10z^2 - \frac{2}{z^2} - 8.$$

This means that $(-1, 1, 1, -1, e) \notin \Sigma(8_4^3)$ and hence that $\Sigma(8_4^3)$ is generated by $(-1, 1, 1, 1, (12))$ and pure invertibility, as claimed. \square

Claim 6.9. *The symmetry group $\Sigma(8_5^3) < \Gamma_3$ is the four element group isomorphic to D_2 generated by $(1, 1, -1, -1, e)$ and $(1, -1, 1, 1, (23))$.*

Proof. Figures 49 and 50 imply that the four element group given above is a subgroup of $\Sigma(8_5^3)$. Now the linking matrix for 8_5^3 is

	1	2	3
1	0	0	0
2	0	0	-1
3	0	-1	0

which is in the standard form $(a, 0, 0)$. This means that the symmetry group must be a subgroup of the 16 element preimage of $\{\{+1\} \times \mathbf{Z}_2 \times \mathbf{Z}_2 \times \{e, (23)\}\}$ in Γ_3 . Computing this preimage, we observe next that if we apply any of the group elements

$$\begin{array}{cccc} (-1, -1, -1, 1, (23)) & (-1, -1, 1, -1, (23)) & (-1, 1, -1, 1, (23)) & (-1, 1, 1, -1, (23)) \\ (-1, -1, -1, 1, e) & (-1, -1, 1, -1, e) & (-1, 1, -1, 1, e) & (-1, 1, 1, -1, e) \end{array}$$

in this preimage to our link, we get a link with Jones polynomial

$$-\frac{1}{z^5} + \frac{3}{z^4} - z^3 - \frac{3}{z^3} + 3z^2 + \frac{6}{z^2} - 4z - \frac{5}{z} + 6$$

while the base link has Jones polynomial

$$-\frac{1}{z^6} + \frac{3}{z^5} - \frac{4}{z^4} + \frac{6}{z^3} - z^2 - \frac{5}{z^2} + 3z + \frac{6}{z} - 3.$$

This rules out those 8 elements, leaving us with a subgroup of 8 possible elements remaining. We claim that among these, the pure exchange $(1, 1, 1, 1, (23))$ is actually ruled out. This claim completes the proof, since any proper subgroup of the 8 element group of remaining elements must have order at most 4, demonstrating that the four element subgroup of $\Sigma(8_5^3)$ generated by the isotopies in Figures 49 and 50 must be the entire group.

Detecting that $(1, 1, 1, 1, (23))$ is ruled out will require us to use the Satellite Lemma (4.3). Figure 3 gives us two satellites of 8_5^3 : one which replaces the second component with a Hopf link and one which replaces

the third component with a Hopf link. By the lemma, if the pure exchange is in the symmetry group, these satellites must be isotopic. But taking the Jones polynomial for each, we get the following polynomials:

$$-\frac{3}{z^{9/2}} + \frac{1}{z^{7/2}} - \frac{2}{z^{5/2}} + z^{3/2} - \frac{1}{z^{3/2}} + \frac{1}{z^{21/2}} - \frac{2}{z^{19/2}} + \frac{1}{z^{17/2}} - \frac{3}{z^{13/2}} - 2\sqrt{z} + \frac{1}{\sqrt{z}}$$

and

$$-\frac{2}{z^{9/2}} + z^{7/2} + \frac{1}{z^{7/2}} - 2z^{5/2} - \frac{3}{z^{5/2}} + z^{3/2} + \frac{1}{z^{3/2}} + \frac{1}{z^{17/2}} - \frac{2}{z^{15/2}} + \frac{1}{z^{13/2}} - 2\sqrt{z} - \frac{3}{\sqrt{z}}.$$

Therefore, they are not isotopic and hence $(1, 1, 1, 1, (23)) \notin \Sigma(8_5^3)$, which completes the proof. \square

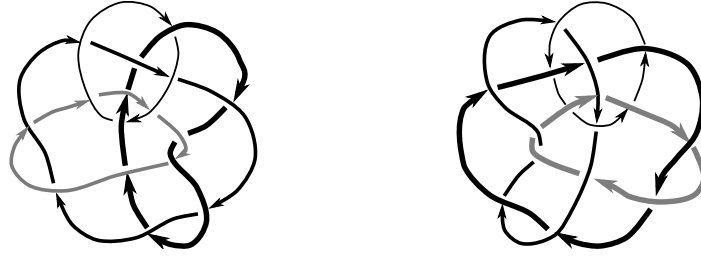


FIGURE 3. Two satellite links, each with companion link 8_5^3 . In the left link, component 2 of 8_5^3 has been replaced by a Hopf link, while in the right link, component 3 of 8_5^3 has been replaced by a Hopf link. Following Lemma 4.3, if the pure exchange symmetry $(1, 1, 1, 1, (23)) \in \Sigma(8_5^3)$, these links must be isotopic. Computing their Jones polynomials shows that they are not.

Claim 6.10. *The subgroup $\Sigma(8_6^3) < \Gamma_3$ is the 8 element group isomorphic to $(\mathbf{Z}_2)^3$ which is generated by $(1, -1, -1, -1, e)$, $(1, -1, 1, 1, (23))$, and $(-1, 1, -1, -1, e)$.*

Proof. Figures 8, 51, and 52, respectively, imply that the three generators above lie in the symmetry group $\Sigma(8_6^3)$. Now the linking matrix for 8_6^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & 0 \\ 3 & -1 & 0 & 0 \end{array}$$

which corresponds to the triple $(0, -1, 1)$ and has the same orbit type as $(a, -a, 0)$. Consulting Table 4, we see that the stabilizer of the linking matrix is precisely the group above, and hence equals $\Sigma(8_6^3)$, which completes the proof. \square

Claim 6.11. *The subgroup $\Sigma(8_7^3) < \Gamma_3$ is the 4 element group isomorphic to D_2 generated by the pure exchange $(1, 1, 1, 1, (23))$ and pure invertibility.*

Proof. Figures 9 and 53 show that $(1, -1, -1, -1, (23))$, $(1, 1, 1, 1, (23)) \in \Sigma(8_7^3)$, which implies that the subgroup above is contained in $\Sigma(8_7^3)$. Now the linking matrix for 8_7^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & -2 \\ 3 & 1 & -2 & 0 \end{array}$$

which is in the standard form (a, b, b) . Consulting Table 4, we see that $|\Sigma(8_7^3)|$ divides 4. Since we already have 4 elements in the group, this completes the proof.

Finally, we note that we have encountered this symmetry group before, as $\Sigma(8_2^3) = \Sigma(8_7^3)$. \square

Claim 6.12. *The subgroup $\Sigma(8_8^3) < \Gamma_3$ is the four element subgroup isomorphic to D_2 generated by pure invertibility and $(1, 1, -1, -1, (23))$.*

Proof. Figures 54 and 55 show that $(1, -1, 1, 1, (23))$, $(1, 1, -1, -1, (23)) \in \Sigma(8_8^3)$, which implies that $\Sigma(8_8^3)$ contains the claimed group. Now the linking matrix for 8_8^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 1 & -1 \\ 2 & 1 & 0 & -2 \\ 3 & -1 & -2 & 0 \end{array}$$

which is in the standard form $(a, b, -b)$. Consulting Table 4, we see that $|\Sigma(8_8^3)|$ divides 4. This completes the proof, since we already have 4 elements in the subgroup.

Finally, we note that we have encountered this symmetry group before, as $\Sigma(8_1^3) = \Sigma(8_8^3)$. \square

Claim 6.13. *The subgroup $\Sigma(8_9^3) < \Gamma_3$ is the 8 element group isomorphic to $(\mathbf{Z}_2)^3$ generated by pure invertibility, the pure exchange $(1, 1, 1, 1, (23))$ and $(1, -1, 1, 1, e)$.*

Proof. Figures 9, 56, and 57 show that $(1, -1, -1, -1, (23))$, $(1, 1, 1, 1, (23))$, $(1, -1, 1, 1, e) \in \Sigma(8_9^3)$, which implies that the 8 element subgroup generated by these elements is a subgroup of $\Sigma(8_9^3)$. Now the linking matrix for this link is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{array}$$

which is in the standard form $(a, 0, 0)$. Consulting Table 4, we see that $|\Sigma(8_9^3)|$ divides 16. Working out these 16 elements as in the case of 8_5^3 , we see that if we apply any of the elements

$$\begin{array}{cccc} (-1, -1, -1, 1, (23)) & (-1, -1, 1, -1, (23)) & (-1, 1, -1, 1, (23)) & (-1, 1, 1, -1, (23)) \\ (-1, -1, -1, 1, e) & (-1, -1, 1, -1, e) & (-1, 1, -1, 1, e) & (-1, 1, 1, -1, e) \end{array}$$

to 8_9^3 , we get a link with Jones polynomial

$$2z^5 - 2z^4 + 4z^3 - 2z^2 + 3z + \frac{1}{z} - 2,$$

while the Jones polynomial of the base 8_9^3 link is

$$z^7 - 2z^6 + 3z^5 - 2z^4 + 4z^3 - 2z^2 + 2z.$$

This leaves only the subgroup claimed. We note that while $\text{Lk}(8_5^3)$ and $\text{Lk}(8_9^3)$ have the same stabilizer, the symmetry group $\Sigma(8_5^3)$ is a proper subgroup of $\Sigma(8_9^3)$. \square

Claim 6.14. *The subgroup $\Sigma(8_{10}^3) < \Gamma_3$ is the 4 element group isomorphic to D_2 generated by pure invertibility and the pure exchange $(1, 1, 1, 1, (12))$.*

Proof. Figures 8 and 58 show that $(1, -1, -1, -1, e)$ and $(1, 1, 1, 1, (12))$ are in $\Sigma(8_{10}^3)$. Now the linking matrix for 8_{10}^3 is

$$\begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 3 & 2 & 2 & 0 \end{array}$$

which has the standard form $(a, a, 0)$. Consulting Table 4, we see that this matrix has an 8 element stabilizer in Γ_3 given by the inverse image of $\langle(1, 1, 1, (12)), (1, 1, -1, e)\rangle$ under the map f . Now we observe that if we apply any of the four elements

$$(-1, -1, -1, 1, (12)) \quad (-1, 1, 1, -1, (12)) \quad (-1, -1, -1, 1, e) \quad (-1, 1, 1, -1, e)$$

in this stabilizer to 8_{10}^3 , we get a link with Jones polynomial

$$z^{10} + z^6 + z^5 + z^3,$$

while the base 8_{10}^3 link has Jones polynomial

$$z^9 + z^7 + z^6 + z^2.$$

This rules out all but the four element subgroup above, completing the proof. \square

7. ISOTOPIES FOR FOUR-COMPONENT LINKS

There are three prime four-component links with 8 crossings. They are quite similar in appearance, with only some crossing changes distinguishing them. Their symmetry computations are made somewhat more difficult by the fact that we are working in the 768 element group Γ_4 . All three of these links are composed of four unknots linked together so that component 1 is linked to 2 and 3, and 2 and 3 are linked to 4.

Here are the symmetry groups for these links, listed in terms of generators for each group. Again, we denote the purely invertible symmetry, i.e., element $(1, -1, -1, -1, -1, e)$, by PI; all three links admit this symmetry.

Link(s)	$ \Sigma(L) $	$\Sigma(L)$ isomorphic to	Generators
8_1^4 (8a21)	16	$\mathbf{Z}_2 \times D_4$	PI, $(1, 1, 1, 1, 1, (23))$, $(1, 1, 1, 1, 1, (1243))$
8_2^4 (8n7)	16	D_8	$(1, 1, 1, 1, 1, (13)(24))$, $(1, 1, 1, -1, 1, (1243))$
8_3^4 (8n8)	32	$\mathbf{Z}_2 \times \mathbf{Z}_2 \times D_4$	PI, $(-1, 1, 1, 1, 1, (23))$, $(-1, 1, 1, 1, 1, (1243))$, $(-1, -1, 1, 1, -1, e)$

TABLE 12. Four-component link symmetry groups, by crossing number

Our approach mimics the one used for three-component links. After calculating the linking matrix, we utilize Table 5 to determine the stabilizer of this matrix within Γ_4 ; we know that the symmetry group $\Sigma(L)$ must be a subgroup of this stabilizer. Next, we show some elements are in the symmetry group by exhibiting isotopies and rule others out using polynomial invariants until we can discern the symmetry group.

Claim 7.1. *The symmetry subgroup for 8_1^4 is the 16 element group isomorphic to $\mathbf{Z}_2 \times D_4$ given by the S^3 -orientation-preserving elements of the inverse image $f^{-1}(S(a, a, a, a))$.*

Proof. The linking matrix for 8_1^4 is

$$\begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 1 \\ 4 & 0 & 1 & 1 & 0 \end{array}$$

which corresponds to the standard form (a, a, a, a) . By Lemma 3.11, $\Sigma(8_1^4)$ must be a subgroup of the 32 element stabilizer $f^{-1}(S(a, a, a, a))$ of the linking matrix.

Figures 59, 60, and 7 show that pure invertibility and $(1, -1, -1, -1, -1, (23))$ and $(1, 1, 1, 1, 1, (1342))$ are all in $\Sigma(8_1^4)$. Together, these generate the 16 element group of the claim.

We must show that the 16 S^3 -orientation-reversing elements of the stabilizer, (i.e., elements of the form $(-1, \epsilon_1, \dots, \epsilon_4, p)$) are not in the symmetry group. If we apply any of these 16 elements to the base link, we obtain a link with Jones polynomial

$$-5z^{9/2} + z^{7/2} - z^{5/2} - z^{21/2} + 3z^{19/2} - 6z^{17/2} + 4z^{15/2} - 7z^{13/2} + 4z^{11/2}.$$

But the Jones polynomial of the base 8_1^4 is

$$4z^{9/2} - 6z^{7/2} + 3z^{5/2} - z^{3/2} - z^{19/2} + z^{17/2} - 5z^{15/2} + 4z^{13/2} - 7z^{11/2},$$

This rules out all 16 of these remaining elements, which proves the claim. \square

Claim 7.2. *The symmetry subgroup for 8_2^4 is the 16 element group isomorphic to D_8 given by the S^3 -orientation-preserving elements of $f^{-1}(S(a, a, -a, a))$.*

Proof. The linking matrix for 8_2^4 is

	1	2	3	4
1	0	1	-1	0
2	1	0	0	1
3	-1	0	0	1
4	0	1	1	0

which corresponds to the standard form $(a, a, -a, a)$ (remember that the ordering of elements given by (4) is not obvious). By Lemma 3.11 the stabilizer of this linking matrix in Γ_4 is a 32 element subgroup isomorphic to $\mathbf{Z}_2 \times D_8$.

Figures 61 and 62 show that $(1, 1, 1, -1, 1, (14))$ and $(1, 1, 1, 1, 1, (13)(24))$ are in $\Sigma(8_1^4)$. Together, these generate the 16 element group of the claim.

We must show that the 16 S^3 -orientation-reversing elements of the stabilizer, (i.e., elements of the form $(-1, \epsilon_1, \dots, \epsilon_4, p)$) are not in the symmetry group. If we apply any of these 16 elements to the base link, we obtain a link with Jones polynomial

$$2z^{9/2} - 4z^{7/2} + z^{5/2} - 4z^{3/2} - 3z^{11/2} + \sqrt{z} - \frac{1}{\sqrt{z}},$$

while the base link has Jones polynomial

$$-4z^{9/2} + z^{7/2} - 4z^{5/2} + 2z^{3/2} - z^{13/2} + z^{11/2} - 3\sqrt{z}.$$

This rules out these 16 remaining elements, so the claim is proven. \square

Claim 7.3. *The symmetry subgroup for 8_3^4 is the 32 element group isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times D_4$ given by $f^{-1}(S(a, -a, -a, a))$.*

Proof. The linking matrix for 8_3^4 is

	1	2	3	4
1	0	1	-1	0
2	1	0	0	-1
3	-1	0	0	1
4	0	-1	1	0

which corresponds to the standard form $(a, -a, -a, a)$. By Lemma 3.11 the stabilizer of this linking matrix in Γ_4 is a 32 element subgroup isomorphic to $\mathbf{Z}_2 \times \mathbf{Z}_2 \times D_4$. This group is generated by the isotopies in Figures 63, 64, 65, and 66, which show that pure invertibility, along with elements

$$(-1, 1, 1, 1, 1, (23)), (-1, -1, 1, 1, -1, e), (-1, 1, 1, 1, 1, (1342)) \in \Sigma(8_3^4).$$

\square

8. COMPARISON OF INTRINSIC SYMMETRY GROUPS WITH ORDINARY SYMMETRY GROUPS FOR LINKS

We now compare our results on intrinsic symmetry groups to the existing literature on symmetry groups for links. Henry and Weeks [HW92, Wee93] report $\text{Sym}(L)$ groups for hyperbolic links up to 9 crossings, while Boileau and Zimmerman [BZ87] computed $\text{Sym}(L)$ groups for nonelliptic Montesinos links with up to 11 crossings, and Bonahon and Siebenmann computed $\text{Sym}(L)$ for the Borromean rings link (6_3^3) as an example of their methods in [BS09, Theorem 16.18].

Comparing all this data with ours, we see that

Lemma 8.1. *Among all links of 8 and fewer crossings with known $\text{Sym}(L)$ groups, the Whitten symmetry group $\Sigma(L)$ is not isomorphic to $\text{Sym}(L)$ only for the links in Table 13.*

Link	$\Sigma(L)$	$ \Sigma(L) $	$\text{Sym}(L)$	$ \text{Sym}(L) $
6_3^2	D_2	4	$(\mathbf{Z}_2)^3$	8
8_5^2	\mathbf{Z}_2	2	D_2	4
8_4^2	D_2	4	$(\mathbf{Z}_2)^3$	8
8_6^2	D_2	4	$(\mathbf{Z}_2)^3$	8
8_7^2	D_2	4	$(\mathbf{Z}_2)^3$	8
8_9^2	\mathbf{Z}_2	2	D_2	4
8_{11}^2	\mathbf{Z}_2	2	D_2	4
8_{14}^2	\mathbf{Z}_2	2	D_2	4
8_{16}^2	\mathbf{Z}_2	2	$(\mathbf{Z}_2)^3$	8
8_4^3	D_2	4	D_4	8

TABLE 13. Among links with 8 or fewer crossings with known $\text{Sym}(L)$ groups, precisely 10 have $\Sigma(L) \not\cong \text{Sym}(L)$.

Our results provide some data on symmetry groups of torus links as well.

Lemma 8.2. *For the $(2, 4)$, $(2, 6)$ and $(2, 8)$ torus links $(4_1^2, 6_1^2, \text{ and } 8_1^2)$, we have $\text{Sym}(L) \simeq \Sigma(L) \simeq D_2$. For the Hopf link, we know that $\text{Sym}^+(L) \simeq \Sigma^+(L) \simeq D_2$, but we do not know the orientation reversing part of $\text{Sym}(L)$.*

Proof. Goldsmith computed [Gol82] the “motion groups” $M(S^3, L_{(np, nq)})$ for torus knots and links in S^3 . Combining her Corollary 1.13 and Theorem 3.7, we see that for the (np, nq) torus link $L_{(np, nq)}$, the subgroup Sym^+ of orientation preserving symmetries is homeomorphic to $M(S^3, L_{(np, nq)})$ if $p + q$ is odd, and an index two quotient group of $M(S^3, L_{(np, nq)})$ if $p + q$ is even. The motion group itself is either D_2 or an 8-element quaternion group. But in either case $\text{Sym}^+ \simeq D_2$.

Now the motion group by itself does not provide any information about the orientation reversing elements in Sym . However, any such element in $\text{Sym} \setminus \text{Sym}^+$ would map to an element in $\Sigma(L_{(np, nq)})$ which reversed orientation on S^3 . Since we have already shown that there are no such elements in $\Sigma(L_{(np, nq)})$ for $n = 1, p = 1, q = 2, 3, 4$, we see that for these links $\text{Sym} = \text{Sym}^+ = \Sigma$. \square

By Proposition 2.9, the group $\Sigma(L)$ is the image of $\text{Sym}(L)$ under a homomorphism, and hence a quotient group of $\text{Sym}(L)$. Further, if $\Sigma(L)$ has only orientation-preserving elements (on S^3) then $\text{Sym}(L)$ does as well. Thus we know something about the $\text{Sym}(L)$ groups of all our links; Table 14 summarizes the new information provided by our approach.

Link	$ \text{Sym}(L) $ divisible by	$\text{Sym}(L)$ has quotient
6_1^3	12	D_6
6_3^3	12	D_6
8_6^3	8	$(\mathbf{Z}_2)^3$
8_7^3	4	D_2
8_8^3	4	D_2
8_9^3	8	$(\mathbf{Z}_2)^3$
8_{10}^3	4	D_2

TABLE 14. New results about $\text{Sym}(L)$ symmetry groups: we know from Proposition 2.9 that $\Sigma(L)$ is a quotient group of $\text{Sym}(L)$. Among the links for which we have computed $\Sigma(L)$ are a collection of links for which we can find no information in the literature on $\text{Sym}(L)$. Our results imply that the unknown $\text{Sym}(L)$ groups listed above must have a certain quotient.

9. FUTURE DIRECTIONS

We have now presented explicit computations of the Whitten symmetry groups for all links with 8 and fewer crossings. In all the cases we studied, the most difficult part of the computation was obtaining explicit isotopies to generate the symmetry group; ruling out the remaining elements of Γ_n was generally done by the application of classical invariants. The most difficult of these cases required us to use the “satellite lemma” and study the classical invariants of satellites of our original link.

While we have presented conventional proofs of all our results, we used computer methods extensively in determining the right line of attack for each link – our method was to use the Mathematica package `KnotTheory` to systematically apply all possible Whitten group elements to each link and then check the knot types of the components, the linking matrix, the Jones polynomial, and the HOMFLYPT polynomial in an attempt to distinguish the new link from the original one. We then checked the computer calculations by hand. This automated method clearly cannot compute $\Sigma(L)$, but it does provide a subgroup $\Sigma'(L)$ of $\Gamma(L)$ which is known to contain $\Sigma(L)$. While we do not currently intend to generate isotopies for links with higher crossing number, we intend to present our computationally-obtained $\Sigma'(L)$ groups for 9, 10, and 11 crossing links in a future publication.

Some of the most natural questions about the Whitten symmetry groups remain unanswered by this type of explicit enumeration: which groups can arise as Whitten groups? Does every subgroup of a given Γ_n arise as a $\Sigma(L)$? We have observed 6 different subgroups of Γ_2 , 11 different subgroups of Γ_3 , and 3 different subgroups of Γ_4 so far. This subject is certainly worth further exploration: can one generate a carefully chosen link with a given symmetry group?

10. ACKNOWLEDGEMENTS

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REFERENCES

- [BS09] Francis Bonahon and Laurence Siebenmann, *New geometric splittings of classical knots, and the classification and symmetries of arborescent knots*, <http://almaak.usc.edu/~fbonahon/Research/Preprints/Preprints.html>, 2009.
- [BZ87] Michel Boileau and Bruno Zimmermann, *Symmetries of nonelliptic Montesinos links*, *Math. Ann.* **277** (1987), no. 3, 563–584. MR MR891592 (88h:57004)
- [CD10] Marc Culler and Nathan Dunfield, *Private correspondence*, 2010.
- [Cer98] Corinne Cerf, *Atlas of oriented knots and links*, *Top. Atlas* **3** (1998), no. 2, 1–32, <http://at.yorku.ca/t/a/i/c/31.htm>.
- [CMP11] Jason Cantarella, Matt Mastin, and Jason Parsley, *Symmetries of composite links*, preprint.
- [Con70] J. H. Conway, *An enumeration of knots and links, and some of their algebraic properties*, *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, Pergamon, Oxford, 1970, pp. 329–358. MR MR0258014 (41 #2661)
- [DH91] Helmut Doll and Jim Hoste, *A tabulation of oriented links*, *Math. Comp.* **57** (1991), no. 196, 747–761, With microfiche supplement. MR MR1094946 (92b:57009)
- [Fox62] R. H. Fox, *Some problems in knot theory*, *Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961)*, Prentice-Hall, Englewood Cliffs, N.J., 1962, pp. 168–176. MR MR0140100 (25 #3523)
- [Gol82] D. L. Goldsmith, *Motion of links in the 3-sphere*, *Math. Scand.* **50** (1982), no. 2, 167–205. MR 672923 (83k:57002)
- [Hil86] Jonathan A. Hillman, *Symmetries of knots and links, and invariants of abelian coverings. I*, *Kobe J. Math.* **3** (1986), no. 1, 7–27. MR MR867798 (88f:57005a)
- [HW92] Shawn R. Henry and Jeffrey R. Weeks, *Symmetry groups of hyperbolic knots and links*, *J. Knot Theory Ramifications* **1** (1992), no. 2, 185–201. MR MR1164115 (93e:57007)
- [KS92] Kouzi Kodama and Makoto Sakuma, *Symmetry groups of prime knots up to 10 crossings*, *Knots 90 (Osaka, 1990)*, de Gruyter, Berlin, 1992, pp. 323–340. MR MR1177431 (93h:57012)
- [Mon75] José M. Montesinos, *Surgery on links and double branched covers of S^3* , *Knots, groups, and 3-manifolds (Papers dedicated to the memory of R. H. Fox)*, Princeton Univ. Press, Princeton, N.J., 1975, pp. 227–259. *Ann. of Math. Studies*, No. 84. MR 0380802 (52 #1699)
- [Mur87] Kunio Murasugi, *Jones polynomials and classical conjectures in knot theory. II*, *Math. Proc. Cambridge Philos. Soc.* **102** (1987), no. 2, 317–318. MR MR898151 (88m:57011)
- [Sak89] Makoto Sakuma, *Realization of the symmetry groups of links*, *Transformation groups (Osaka, 1987)*, *Lecture Notes in Math.*, vol. 1375, Springer, Berlin, 1989, pp. 291–306. MR MR1006701 (90m:57008)
- [Thi88] Morwen B. Thistlethwaite, *Kauffman’s polynomial and alternating links*, *Topology* **27** (1988), no. 3, 311–318. MR MR963633 (90c:57005)
- [Wee93] Jeffrey R. Weeks, *Convex hulls and isometries of cusped hyperbolic 3-manifolds*, *Topology Appl.* **52** (1993), no. 2, 127–149. MR MR1241189 (95a:57021)
- [Whi69] W. C. Whitten, Jr., *Symmetries of links*, *Trans. Amer. Math. Soc.* **135** (1969), 213–222. MR MR0242146 (39 #3480)
- [Whi71] Wilbur Whitten, *On prime noninvertible links*, *Bulletin of the Australian Mathematical Society* **5** (1971), no. 1, 127–130.

APPENDIX A. GUIDE TO LINK ISOTOPY FIGURES

The Appendices contain figures representing the 101 isotopies presented in our paper; they are organized as follows.

- Appendix A explains the moves depicted in these isotopies.
- Appendix B exhibits 41 isotopies which can be found by simply rotating about one axis.
- Appendix C contains the remaining isotopies for 2-component links
- Appendix D contains the remaining isotopies for 3-component links
- Appendix E contains the remaining isotopies for 4-component links

For coordinates on our figures, we assume the diagrams appear in the xy -plane with the z -axis coming out of the page. With this orientation, Figure 4 shows symbols for various rotations around lines. We will also denote moves during an isotopy with arrows, as shown in Figure 5.

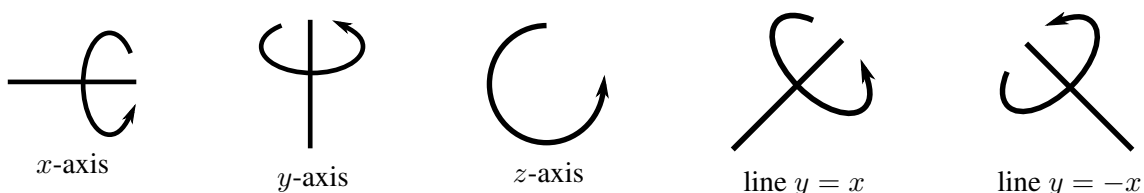


FIGURE 4. Rotation by π radians around various axes is denoted by the symbols above.

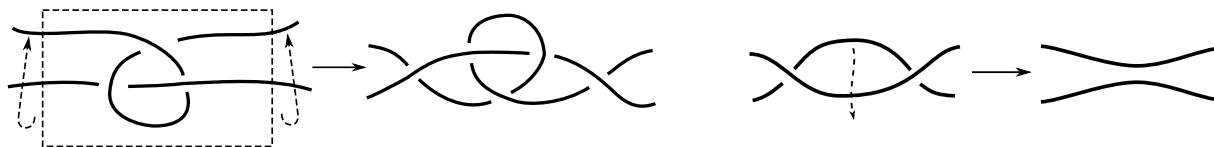


FIGURE 5. The left figure shows a “flip” move. In this move, we take the portion of the link in the dotted box and flip it in the direction indicated by the curved arrows, resulting in the diagram shown on the right. Arrows denote transitions between stages of the isotopy. The right figure shows a simpler transformation. Here the arrow shows a portion of the link. Whether this portion moves over or under intervening portions of the link should be clear from the next drawing.

In all of our diagrams, the component numbered 1 is drawn with a thin (1 pt) line, while other components are denoted by thicker (2, 3, or 4 pt lines). We also number the components explicitly to prevent any confusion on this point. As usual, arrows denote orientation on the link components.

APPENDIX B. ISOTOPY FIGURES FOUND BY ROTATIONS

In this section, we display the simplest type of isotopies: mere rotation of a link about a coordinate axis. Such a rotation suffices for 41 of the 101 isotopies we present in this paper.

We display these in three subsections. The first covers the pure exchange isotopies, which only permute the components; these correspond to Whitten group elements $\gamma = (1, 1, \dots, 1, p)$. The second subsection covers the pure invertibility isotopies, which only invert each components and correspond to $\gamma = (1, -1, -1, \dots, -1, e)$. The third subsection contains 10 other isotopies found by rotating the figure.

B.1. Pure exchange isotopies found by rotation.

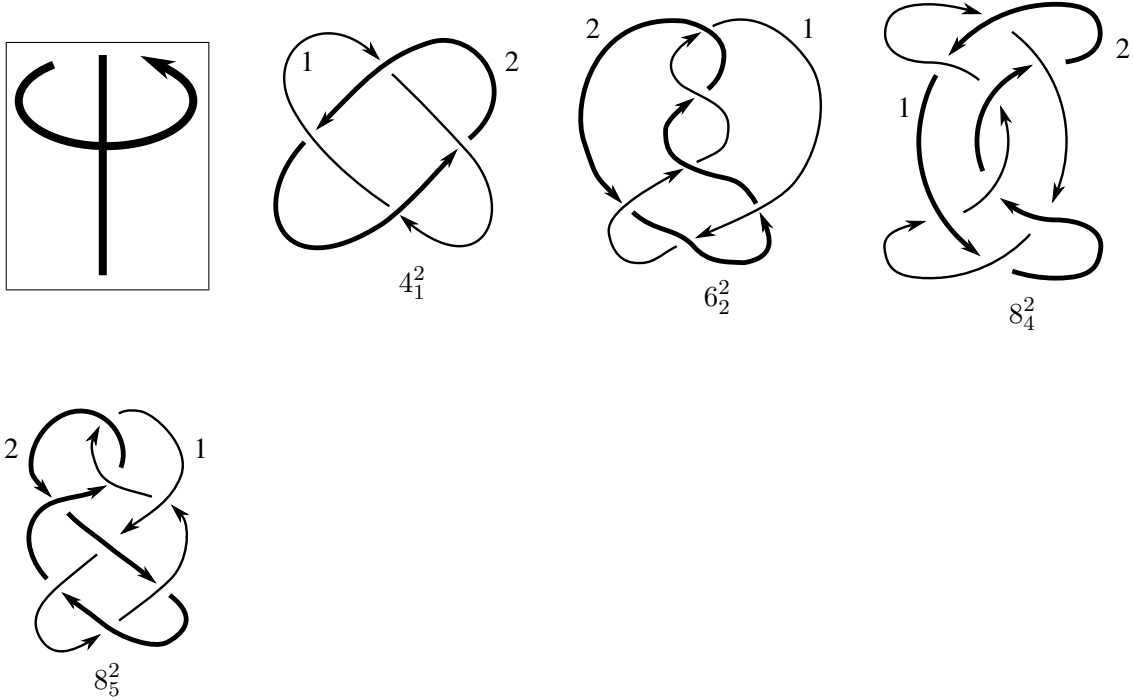


FIGURE 6. Pure exchange isotopies found by rotation about the y -axis.

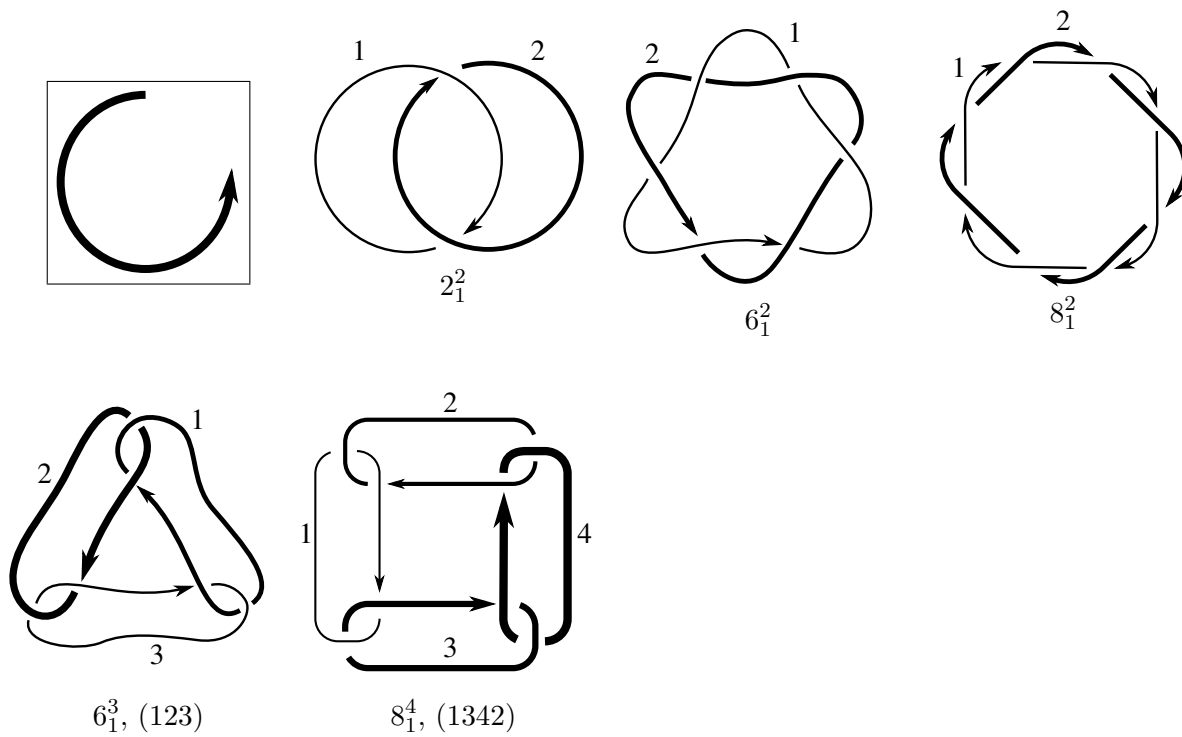
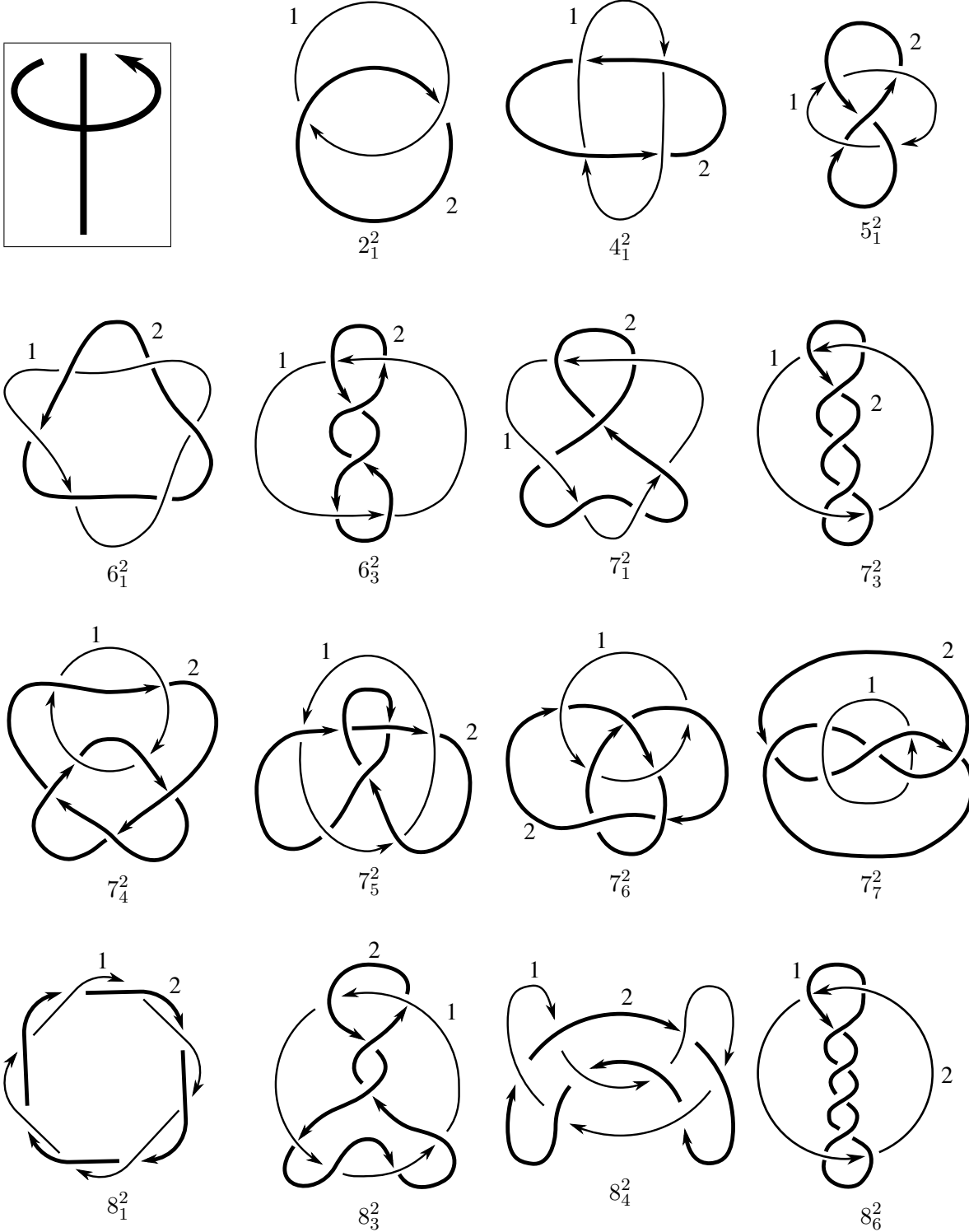


FIGURE 7. Pure exchange isotopies found by rotation about the z -axis. In the case of links with more than two components, all possible exchanges are listed underneath the respective figures.

B.2. Pure invertibility isotopies found by rotation.



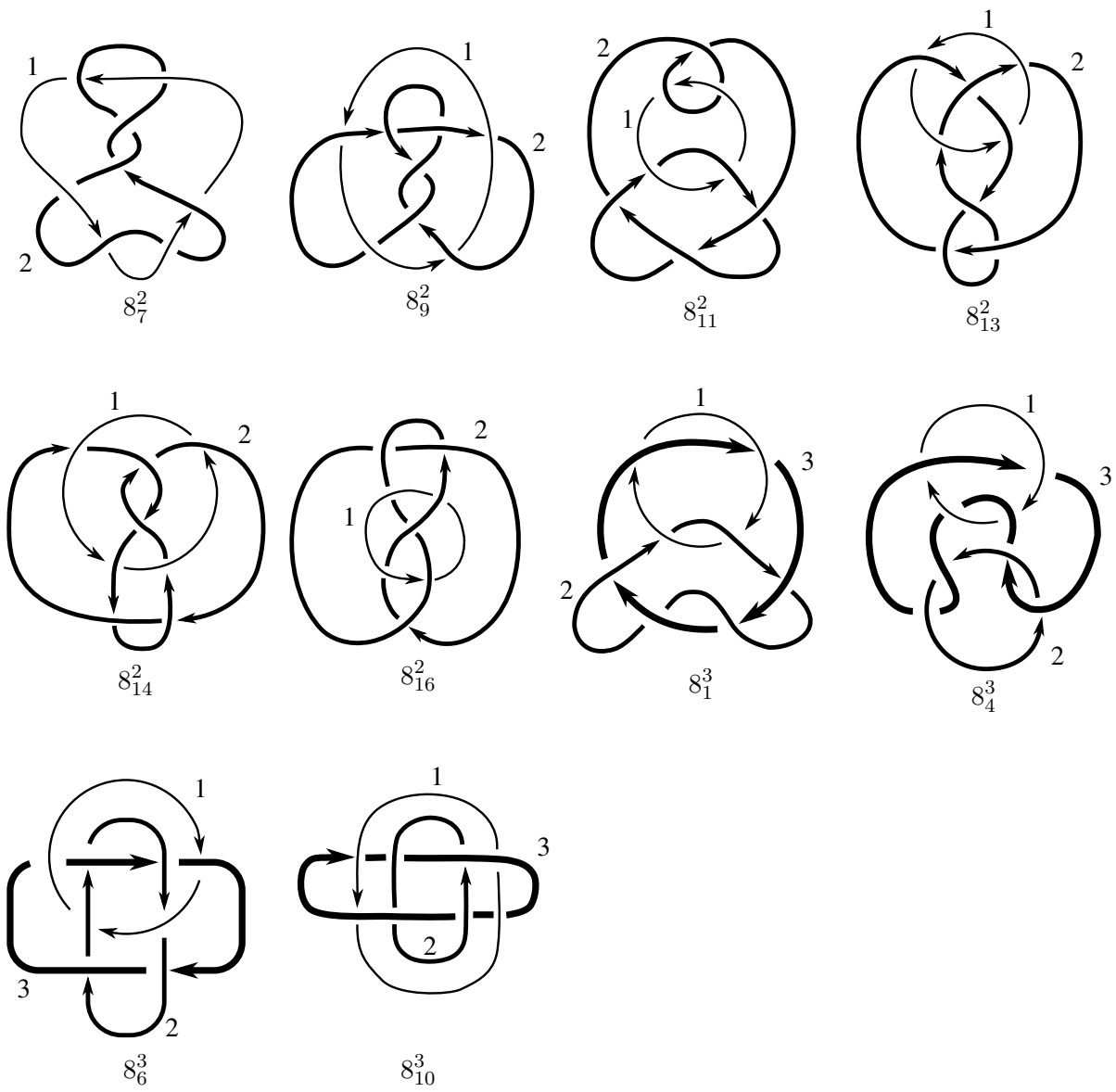


FIGURE 8. Pure inversion isotopies found by rotation about the y -axis.

B.3. Other isotopies found by rotation.

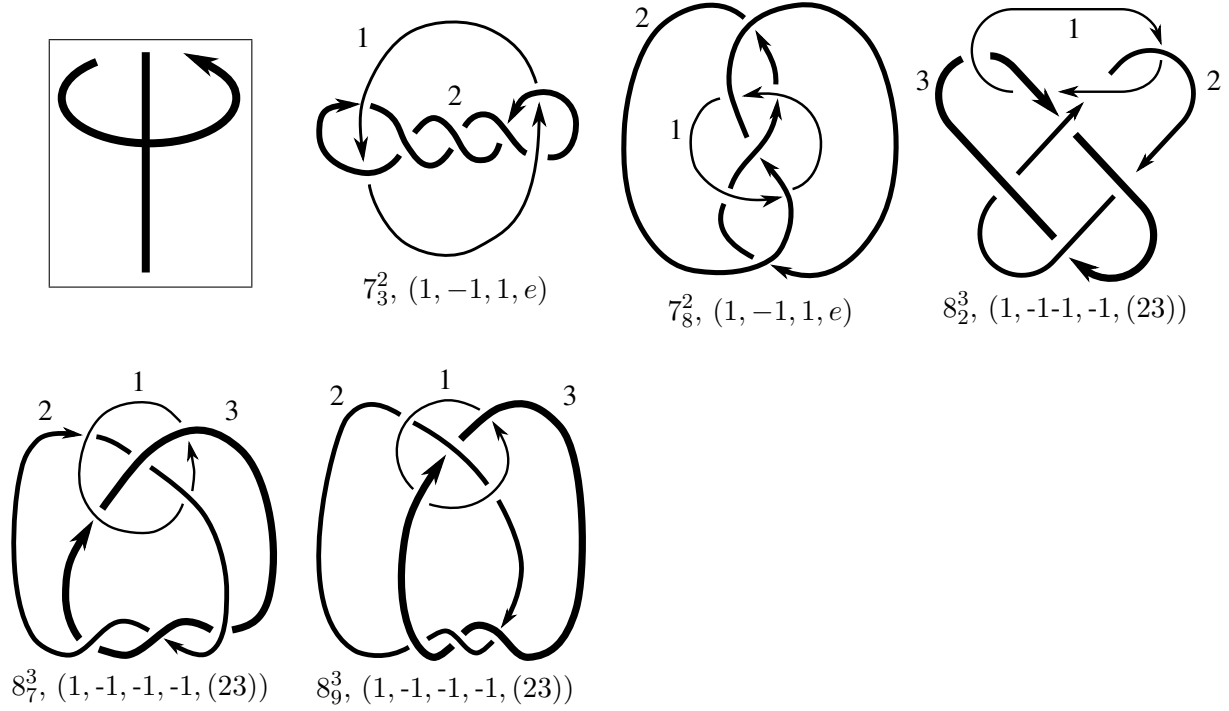


FIGURE 9. Whitten group elements effected through rotation about the y -axis. Group elements are listed under their respective figure.

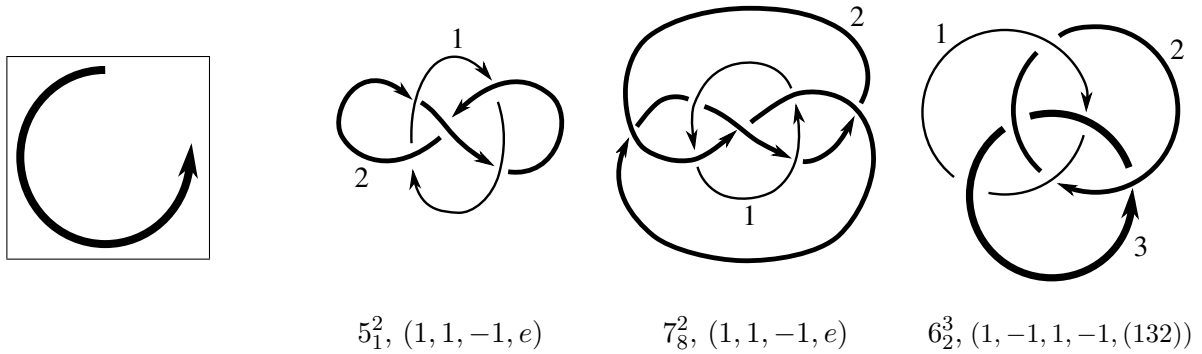


FIGURE 10. Whitten group elements effected through rotation about the z -axis. Group elements are listed under their respective figures.

APPENDIX C. ISOTOPY FIGURES FOR TWO-COMPONENT LINKS

This appendix contains all of the isotopy figures for two-component links that are not contained in Appendix B. Section C.1 contains the pure exchange isotopies. Section C.2 contains the pure exchange isotopies.

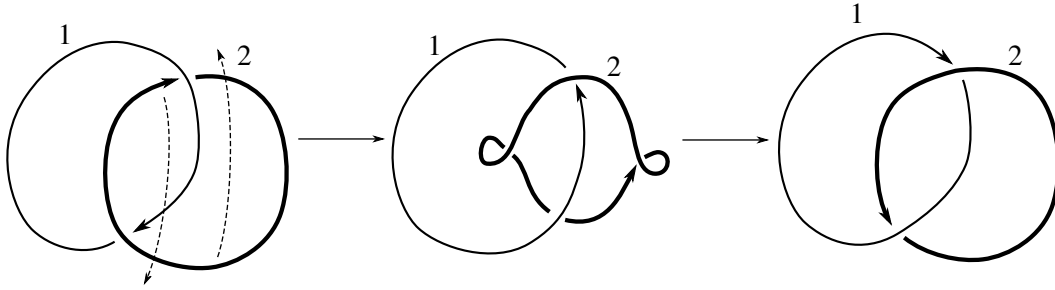


FIGURE 11. $(2_1^2)^\gamma$, $\gamma = (-1, 1, -1, e)$

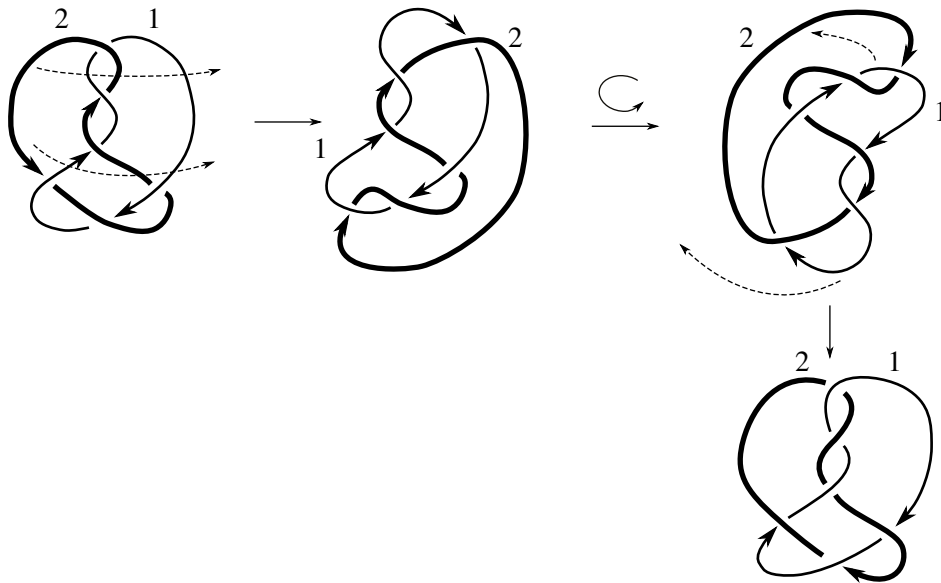


FIGURE 12. $(6_2^2)^\gamma$, $\gamma = (-1, 1, -1, e)$

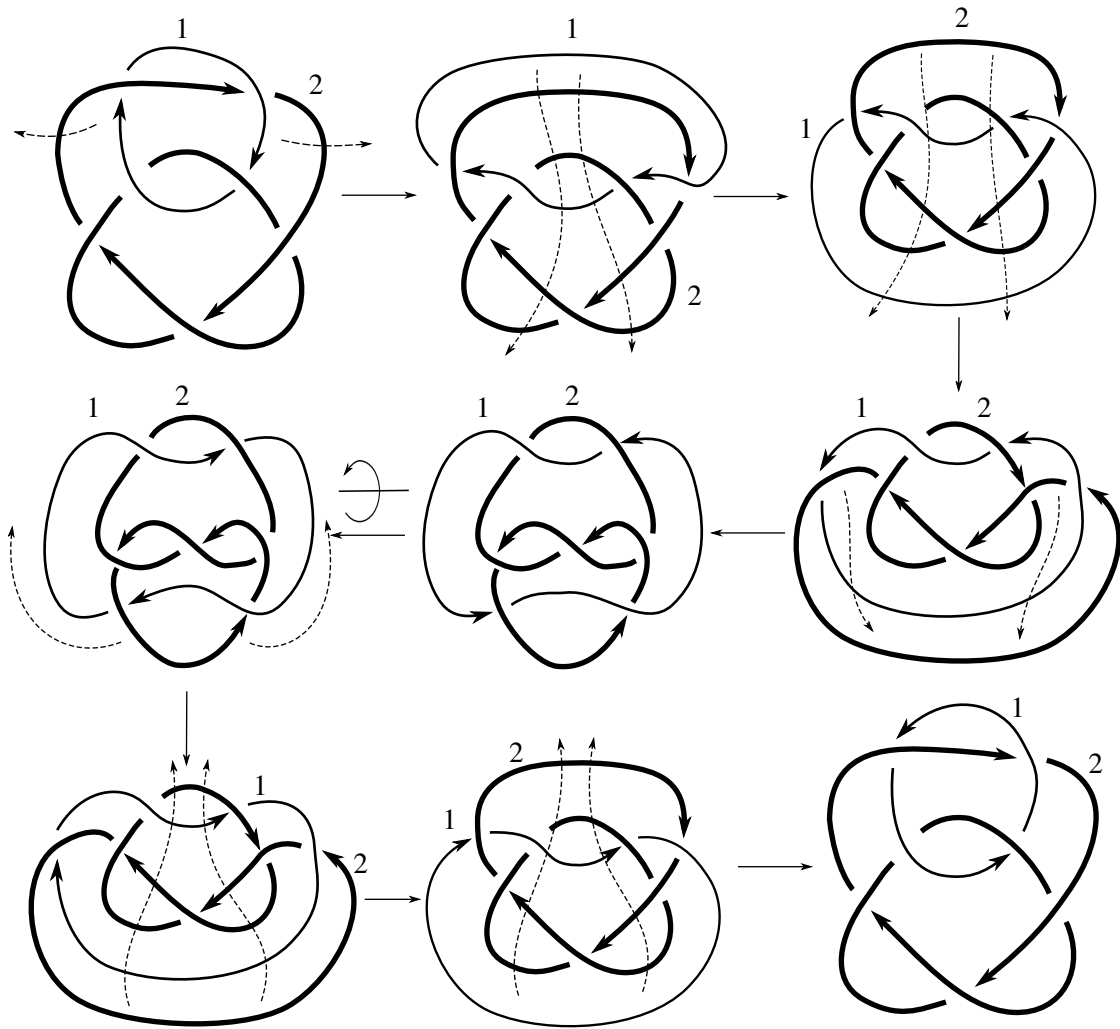


FIGURE 13. $(7_4^2)^\gamma$, $\gamma = (1, -1, 1, e)$

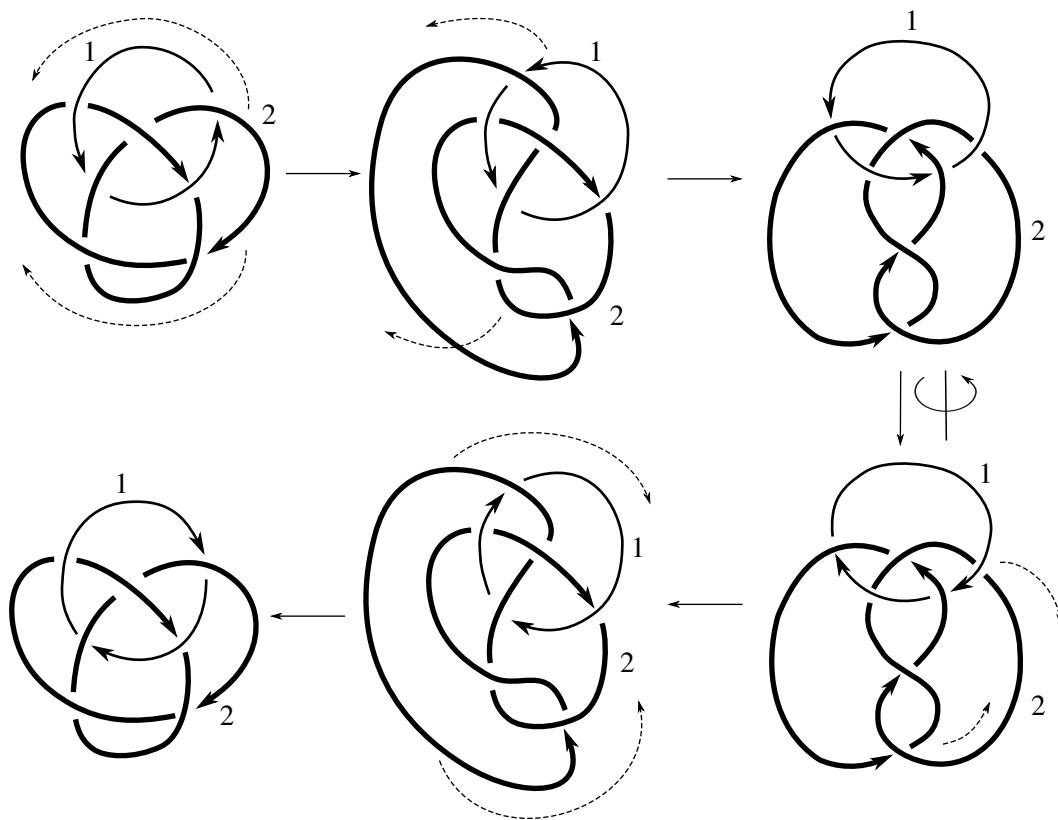


FIGURE 14. $(7_6^2)^\gamma$, $\gamma = (1, -1, 1, e)$

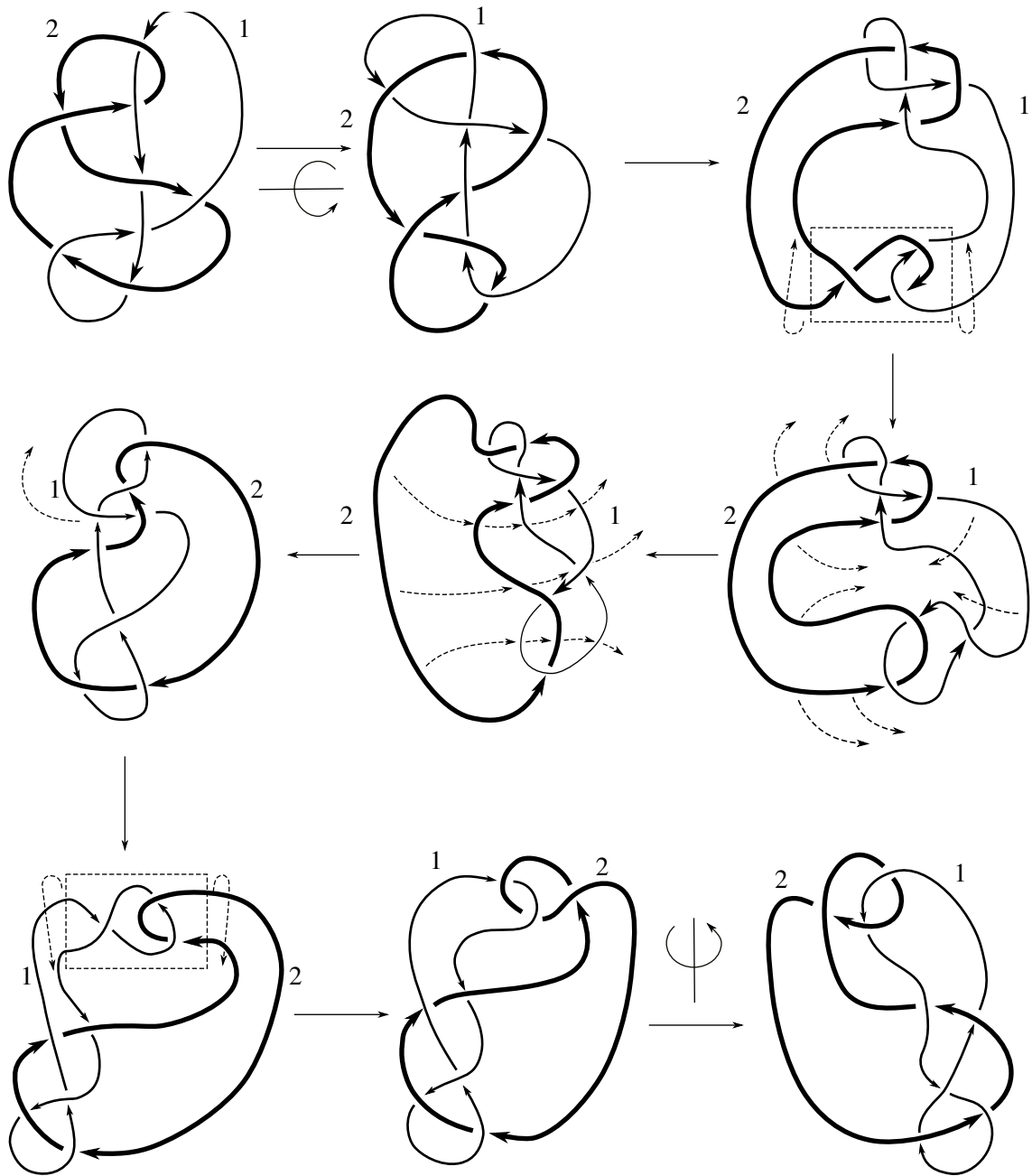


FIGURE 15. $(8_8^2)^\gamma$, $\gamma = (-1, 1, -1, e)$

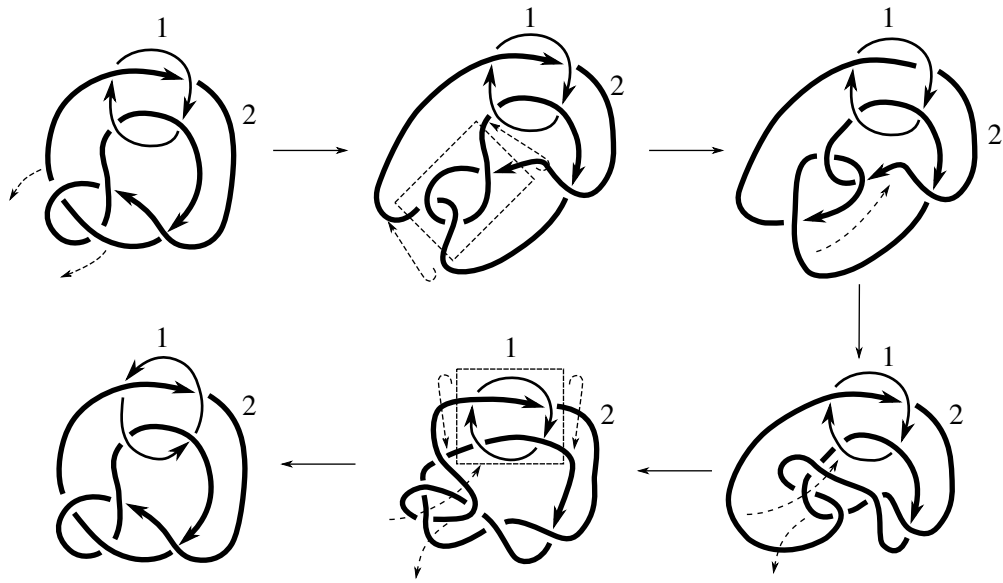


FIGURE 16. $(8_{10}^2)^\gamma$, $\gamma = (1, -1, 1, e)$

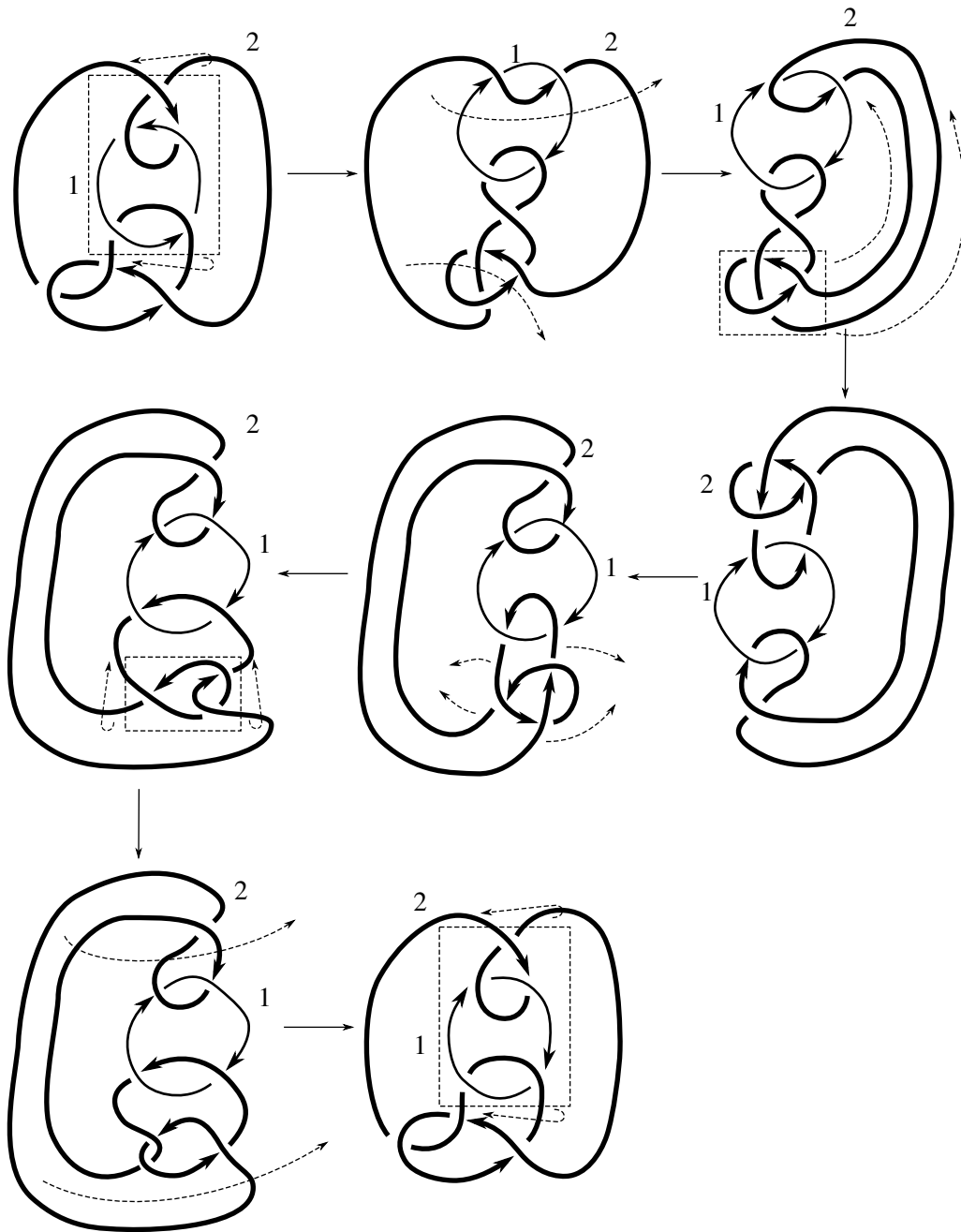


FIGURE 17. $(8_{12}^2)^\gamma$, $\gamma = (1, -1, 1, e)$

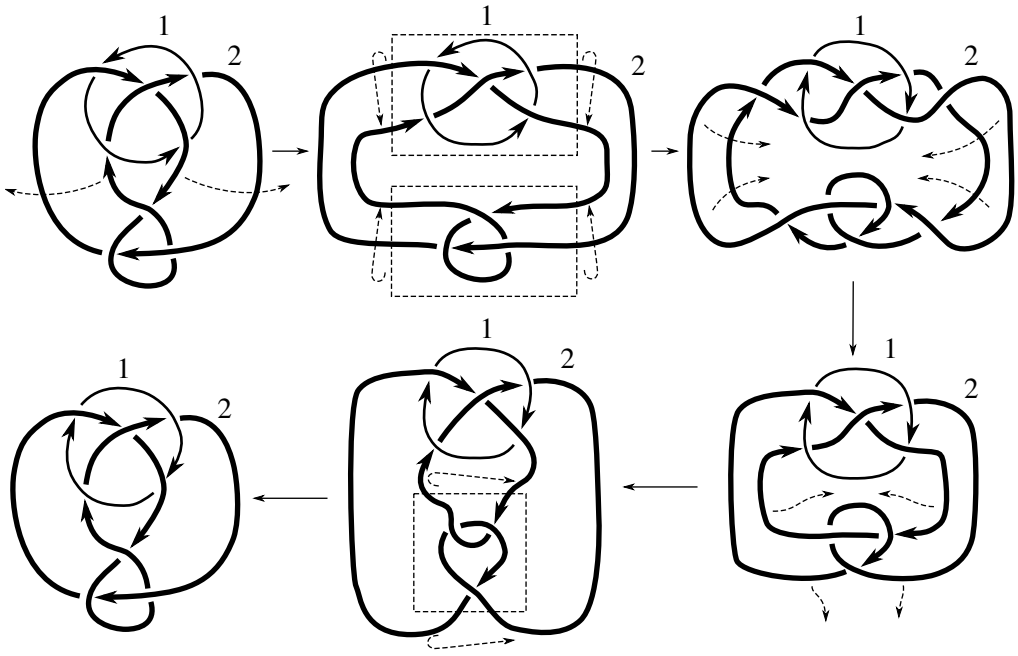


FIGURE 18. $(8_{13}^2)^\gamma$, $\gamma = (1, -1, 1, e)$

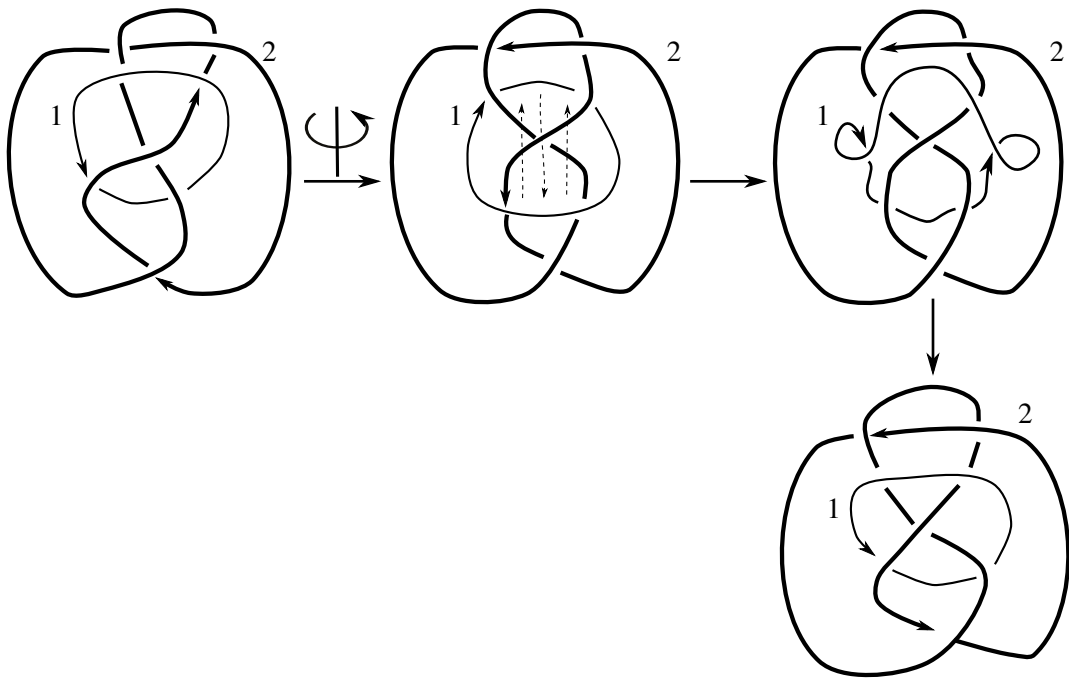


FIGURE 19. $(8_{15}^2)^\gamma$, $\gamma = (1, 1, -1, e)$

C.1. Isotopies showing pure exchange symmetries for two-component links. Along with Appendix B.1, Figures 20-26 demonstrate that each of the 17 two-component links listed in Lemma 5.3 has pure exchange symmetry. We demonstrate in section 5.2 that the 13 remaining two-component links with ≤ 8 crossings do not admit this symmetry.

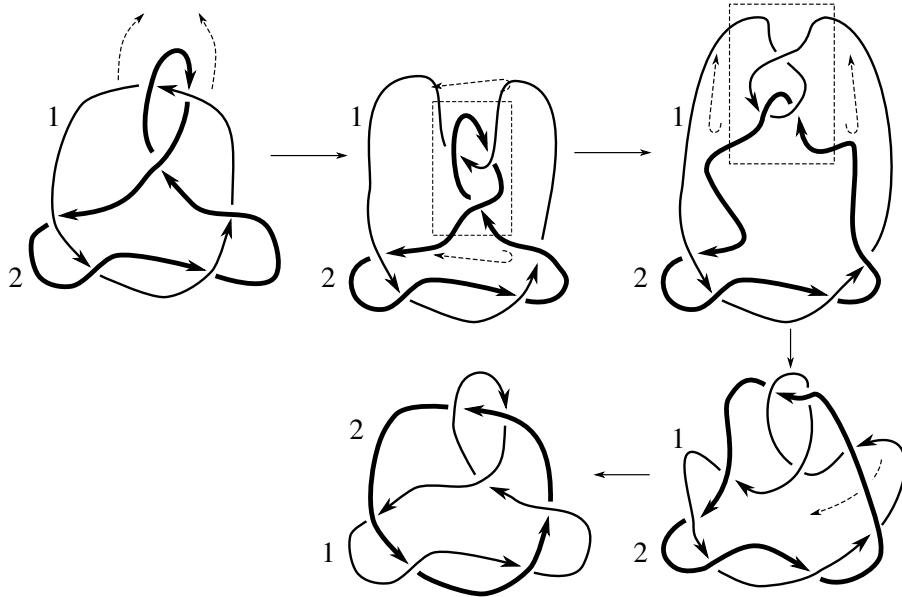


FIGURE 20. $(7_1^2)^\gamma$, $\gamma = (1, 1, 1, (12))$

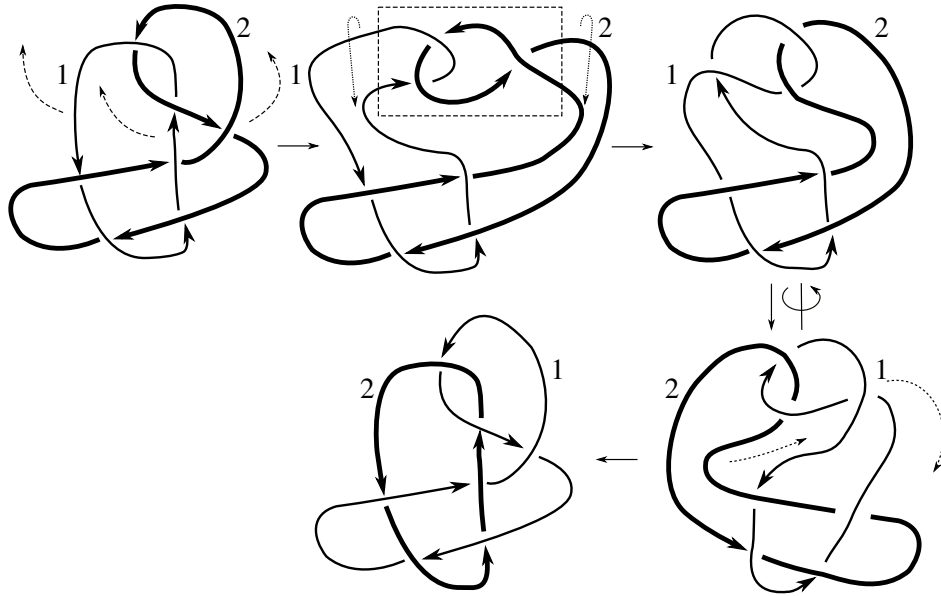


FIGURE 21. $(7_2^2)^\gamma$, $\gamma = (1, 1, 1, (12))$

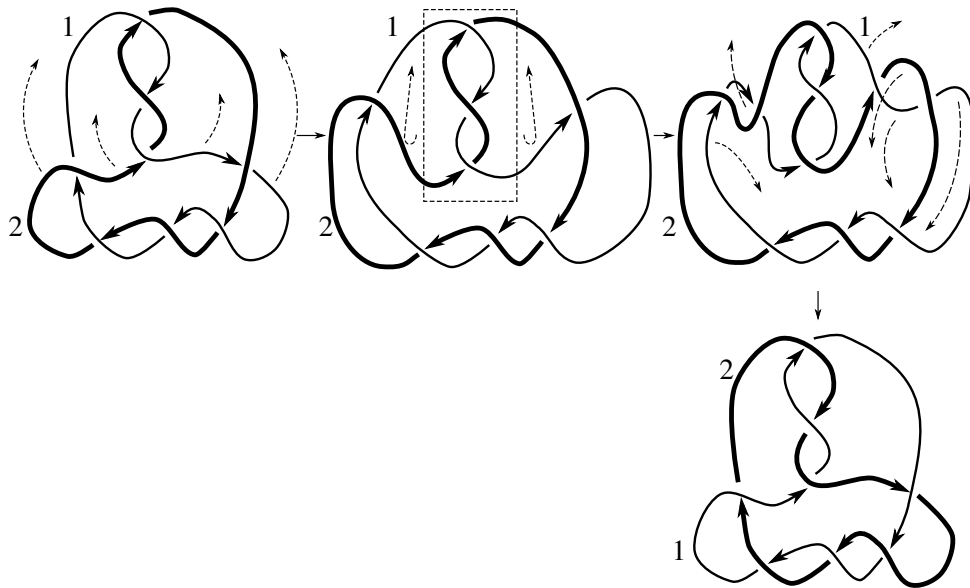


FIGURE 22. $(8_2^2)^\gamma$, $\gamma = (1, 1, 1, (12))$

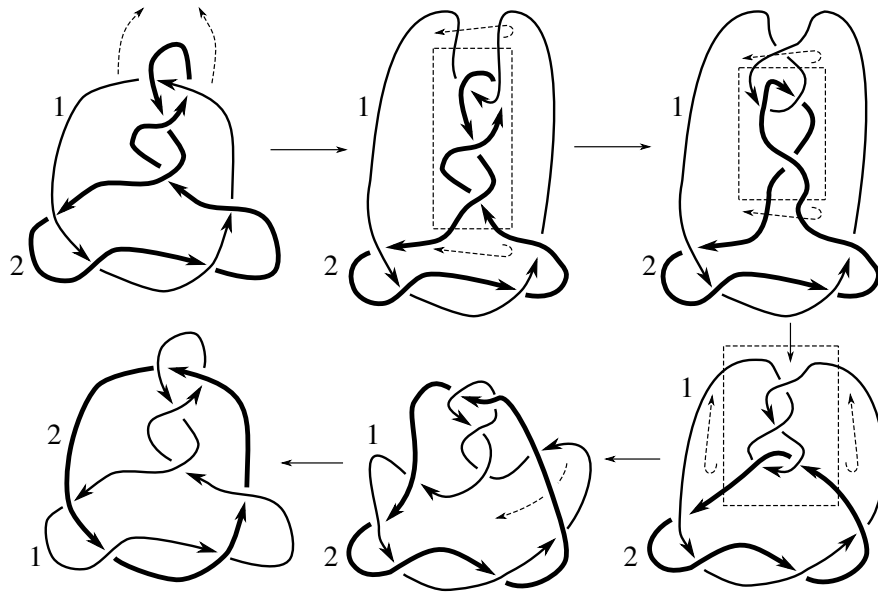


FIGURE 23. $(8_3^2)^\gamma$, $\gamma = (1, 1, 1, (12))$

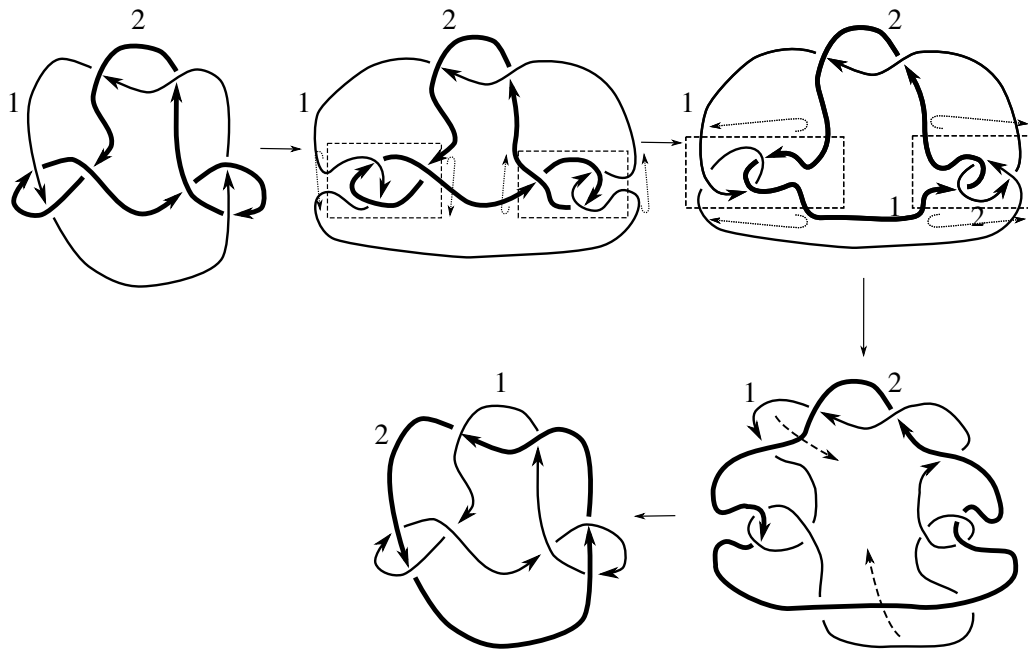


FIGURE 24. $(8_7^2)^\gamma$, $\gamma = (1, 1, 1, (12))$

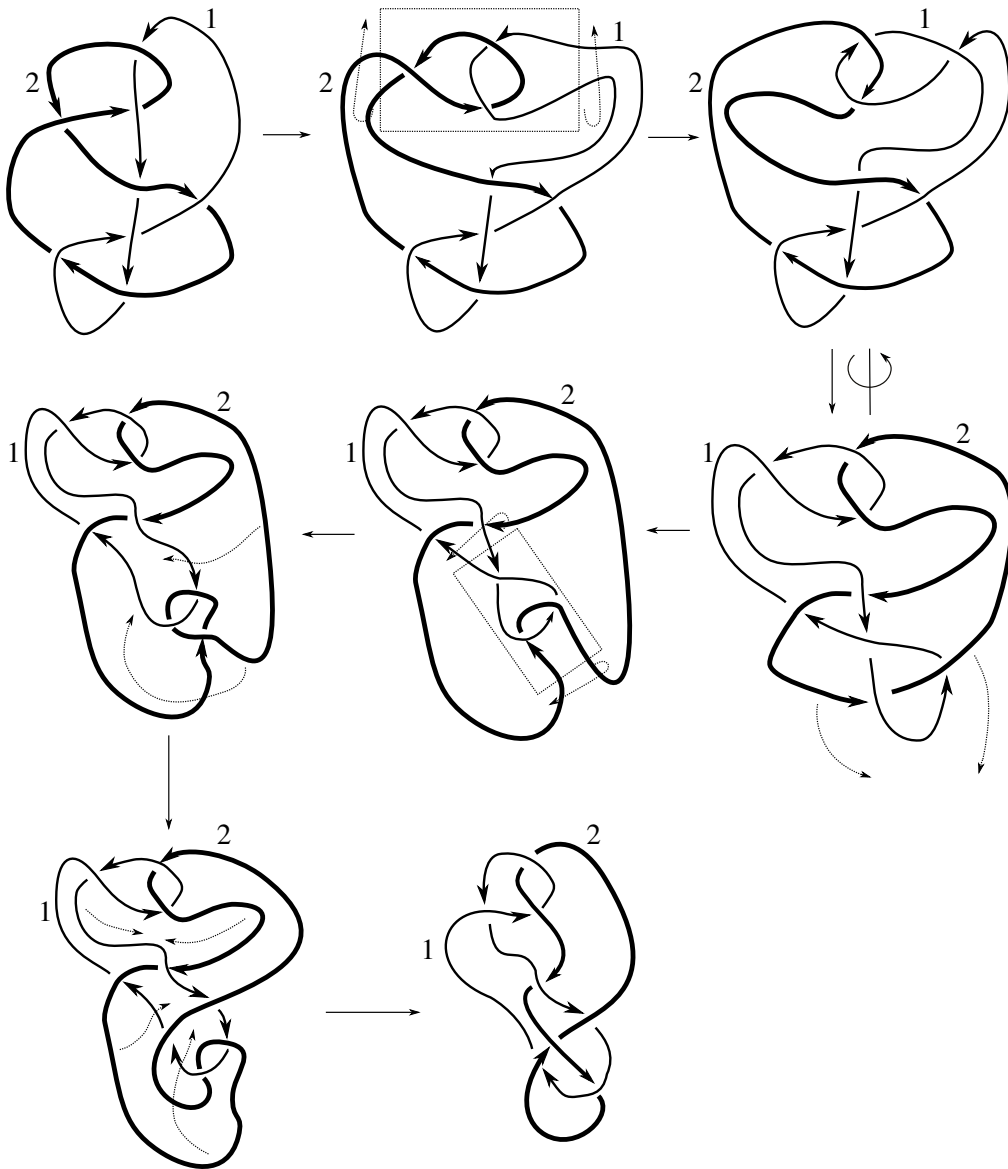


FIGURE 25. $(8_8^2)^\gamma$, $\gamma = (1, 1, 1, (12))$

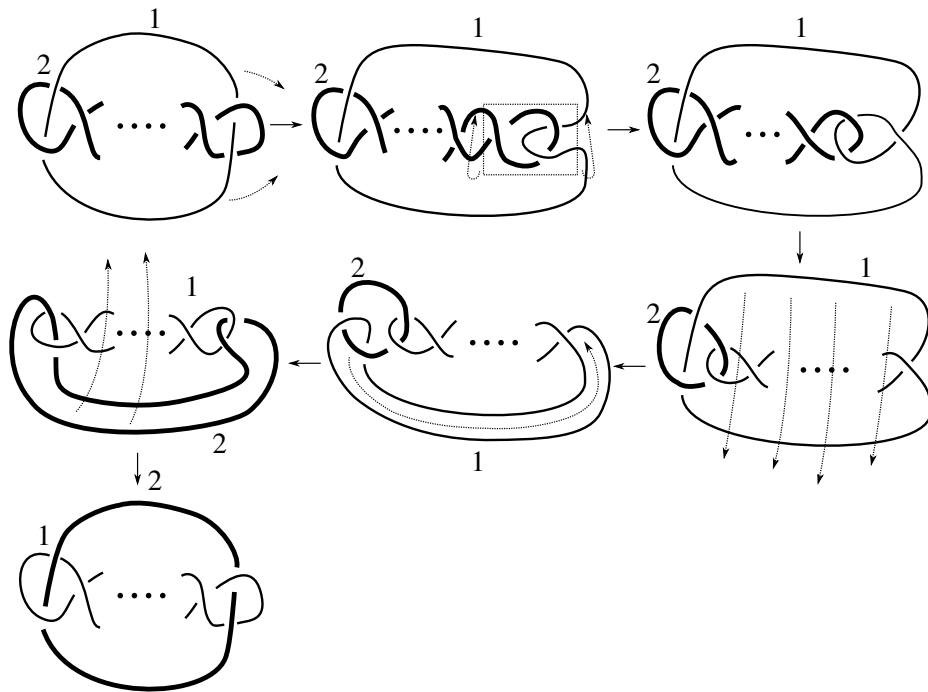


FIGURE 26. The links 5_1^2 , 6_3^2 , 7_3^2 and 8_6^2 share the common form above, with 1, 2, 3, and 4 crossings replacing the dots in the central “twisted” region of component 2 above. The pure exchange symmetry for each can be accomplished in a similar way, by shifting twists from component 2 to component 1 as shown. Since the orientations vary between links, we do not show arrows above. Some examples require a final flip or twist to match orientations, but in each case it is not difficult to figure out the required move.

C.2. Isotopies showing pure invertibility for two-component links. Along with the 21 isotopies exhibited in Appendix B.2, Figures 27-34 demonstrate that, as stated in Lemma 5.4, all 30 of the two-component links with eight or fewer crossings are purely invertible. (To obtain invertibility for 7_8^2 , combine the results of Figures 9 and 10, which show that each of its components can be individually inverted.)

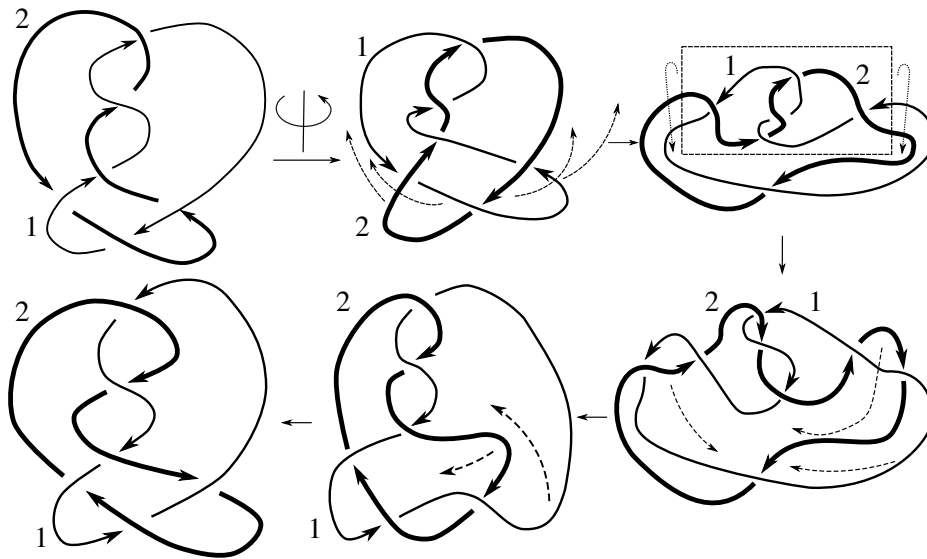


FIGURE 27. $(6_2^2)^\gamma$, $\gamma = (1, -1, -1, e)$

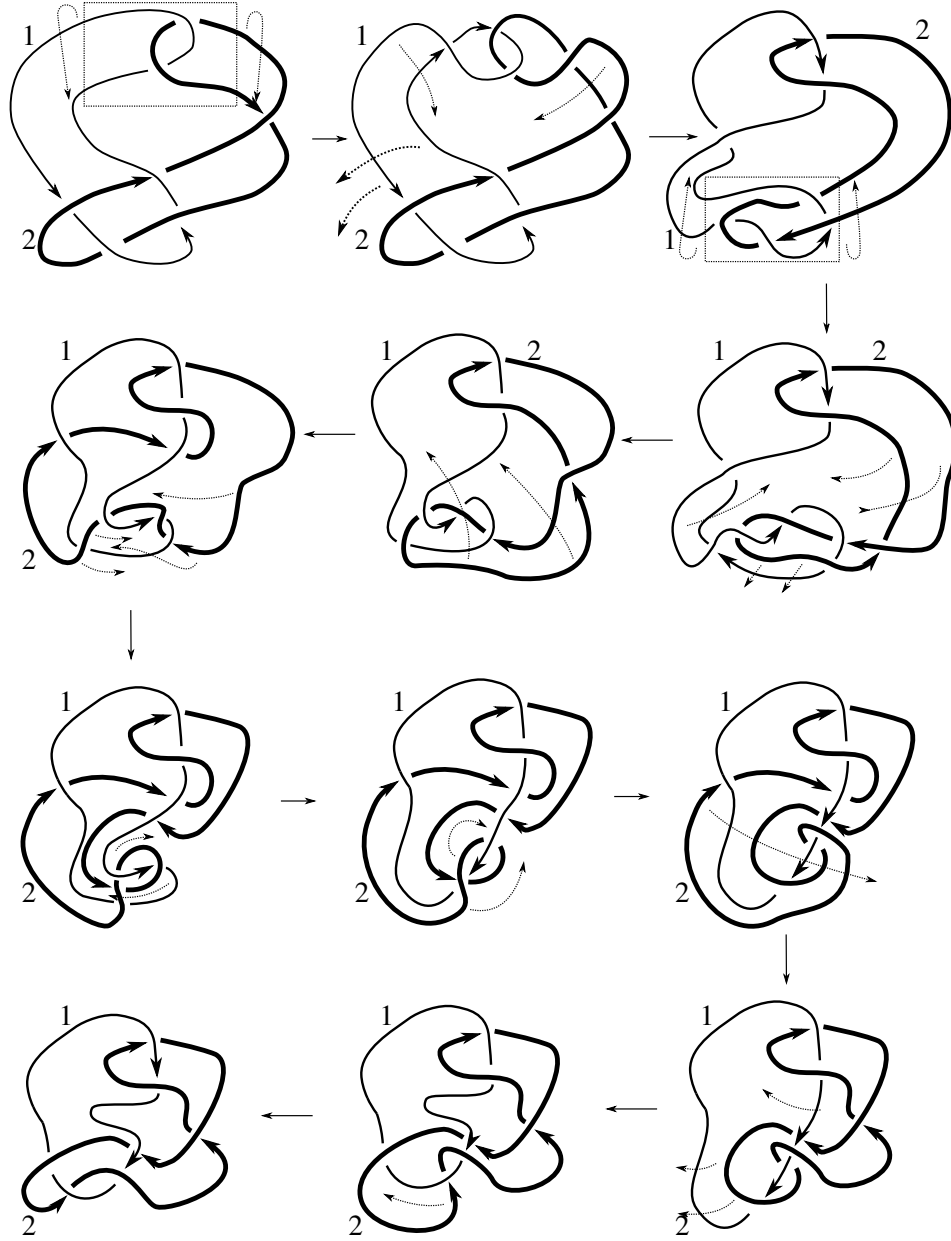


FIGURE 28. $(7_2^2)^\gamma$, $\gamma = (1, -1, -1, e)$

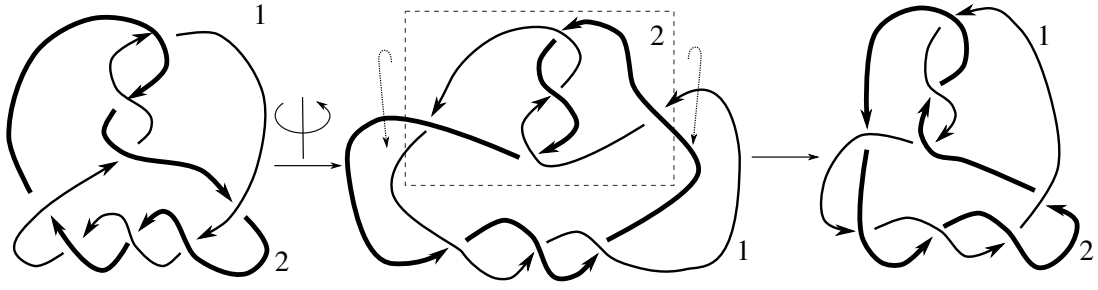


FIGURE 29. $(8_2^2)^\gamma$, $\gamma = (1, -1, -1, e)$

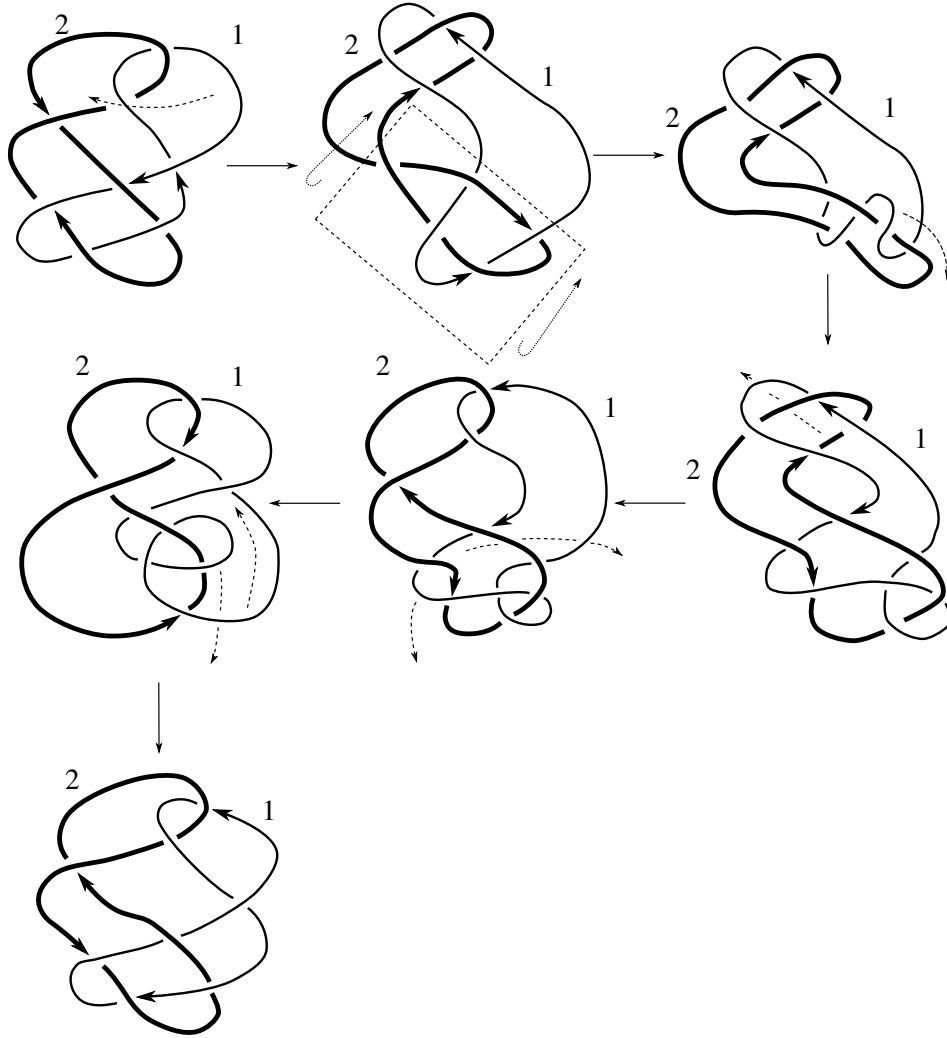


FIGURE 30. $(8_5^2)^\gamma$, $\gamma = (1, -1, -1, e)$

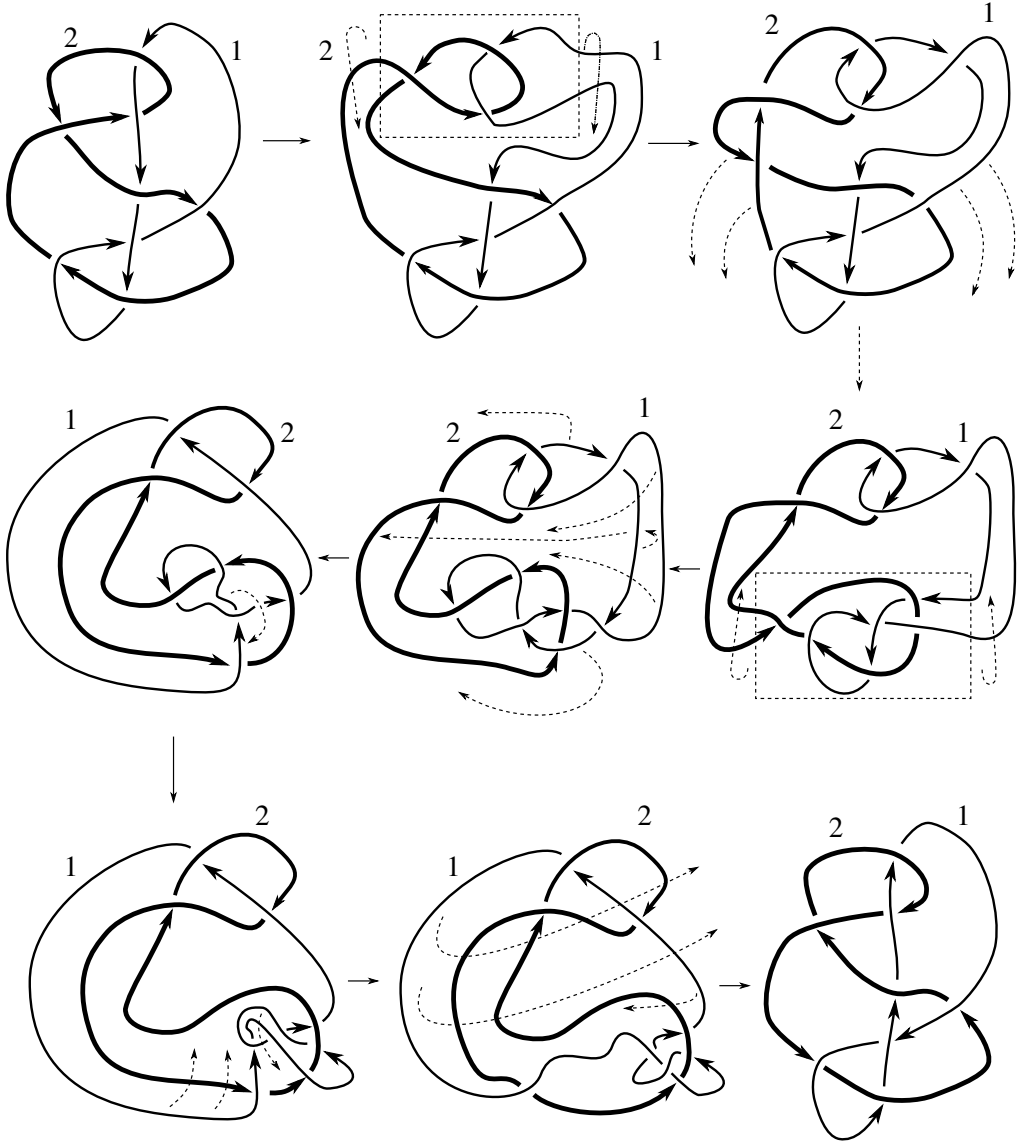


FIGURE 31. $(8_8^2)^\gamma$, $\gamma = (1, -1, -1, e)$

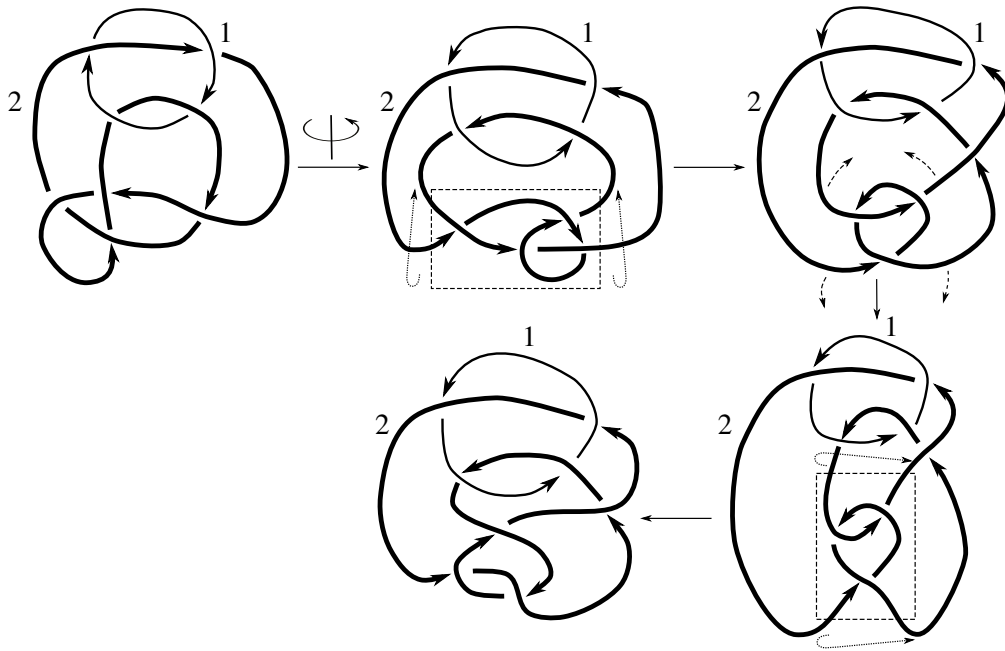


FIGURE 32. $(8_{10}^2)^\gamma$, $\gamma = (1, -1, -1, e)$

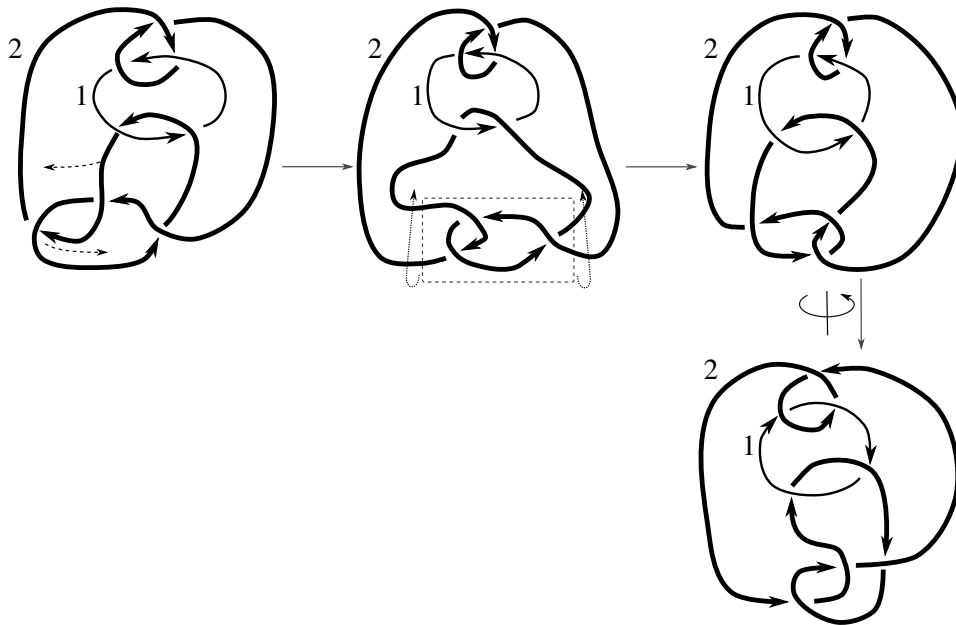


FIGURE 33. $(8_{12}^2)^\gamma$, $\gamma = (1, -1, -1, e)$

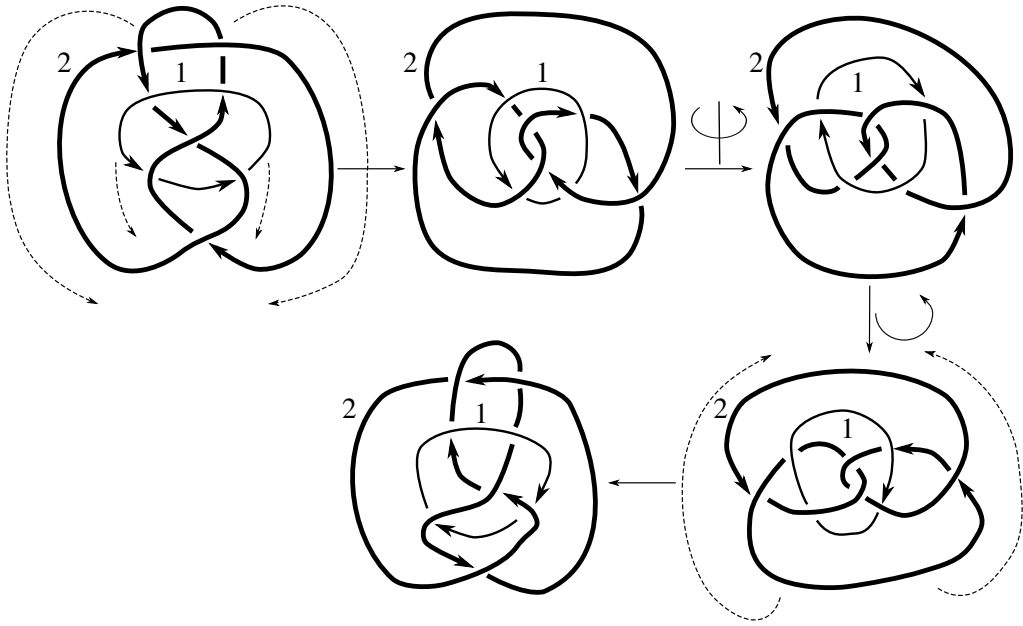


FIGURE 34. $(8_{15}^2)^\gamma$, $\gamma = (1, -1, -1, e)$

APPENDIX D. ISOTOPY FIGURES FOR THREE-COMPONENT LINKS

This appendix contains all of the isotopy figures for three-component links that are not contained in Appendix B.

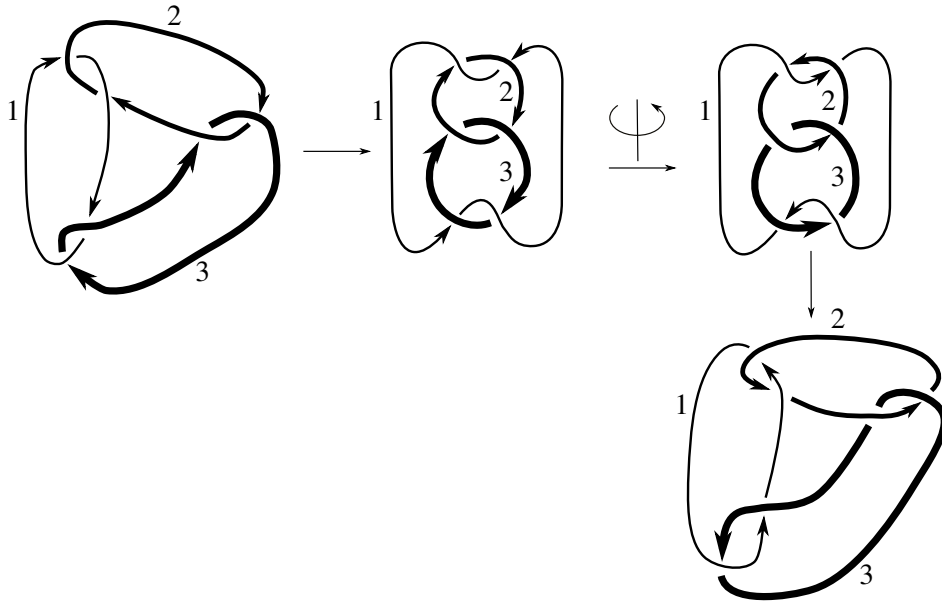


FIGURE 35. $(6_1^3)^\gamma$, $\gamma = (1, -1, -1, -1, e)$

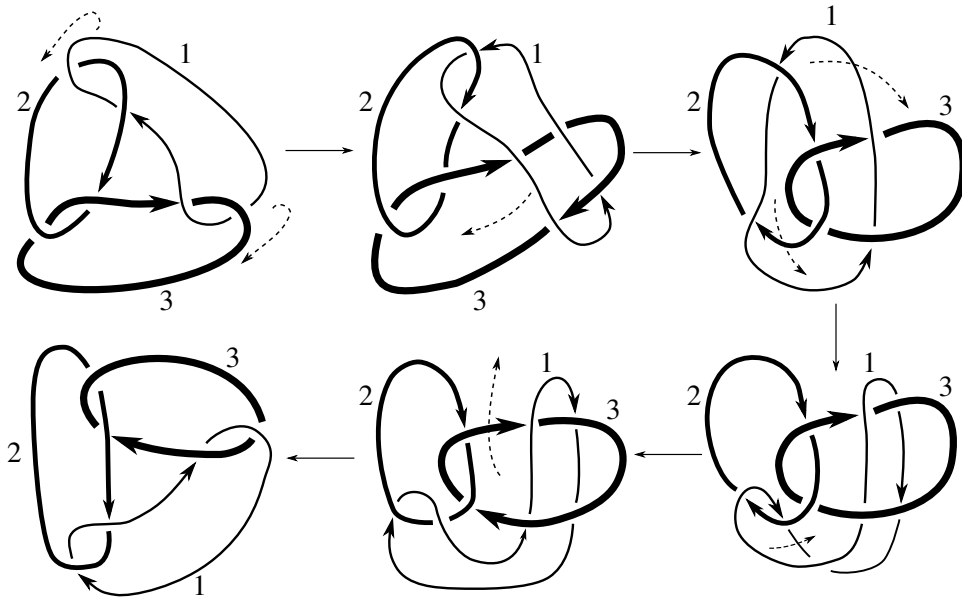


FIGURE 36. $(6_1^3)^\gamma$, $\gamma = (1, 1, 1, 1, (23))$

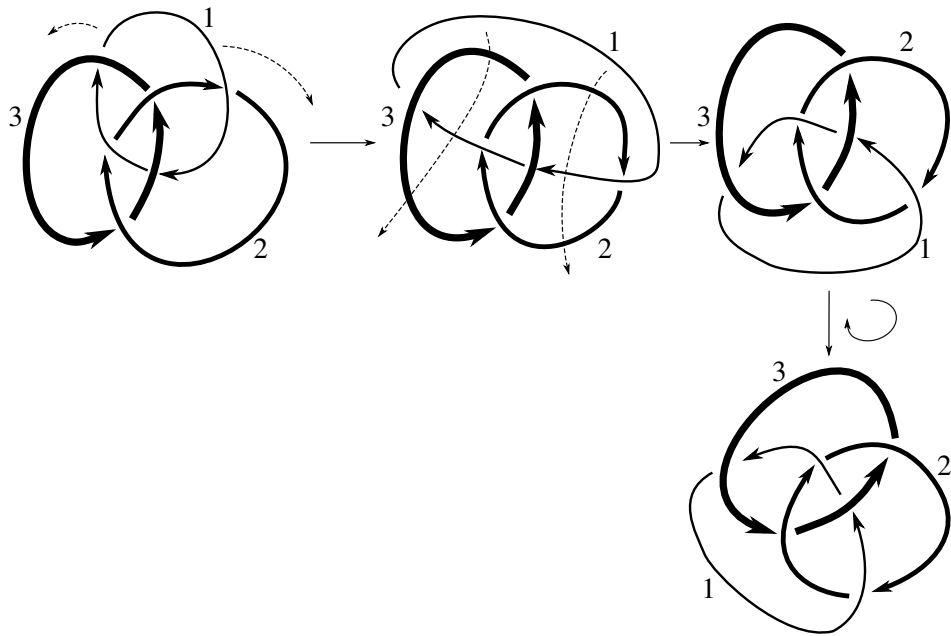


FIGURE 37. $(6_2^3)^\gamma$, $\gamma = (-1, -1, 1, 1, (13))$

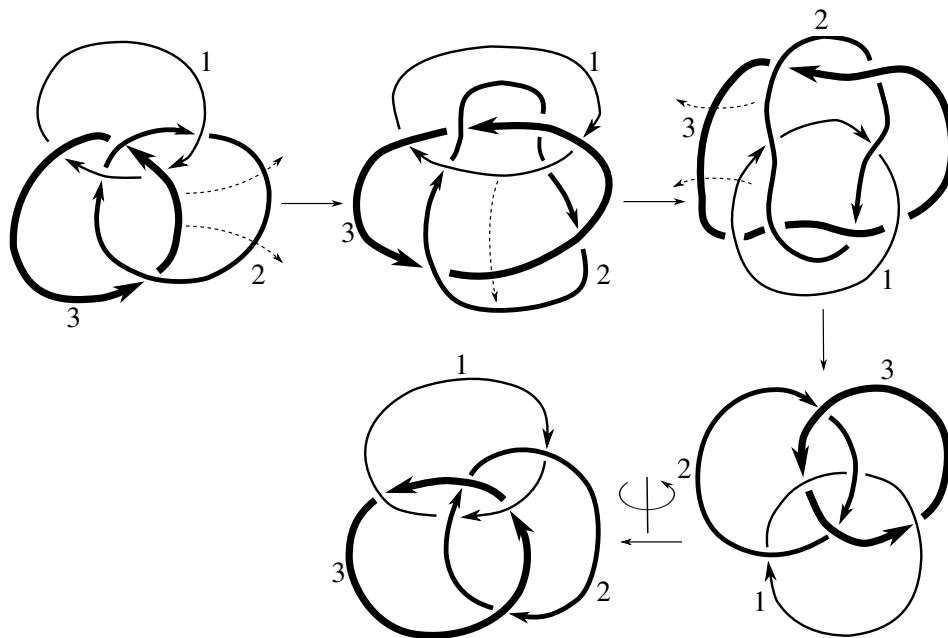


FIGURE 38. $(6_2^3)^\gamma$, $\gamma = (-1, 1, 1, 1, e)$

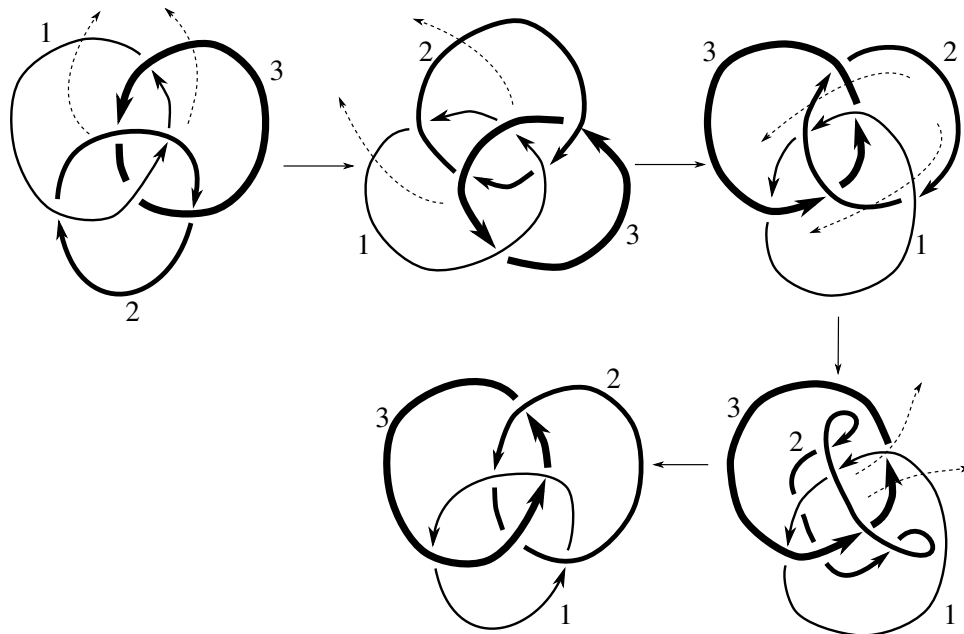


FIGURE 39. $(6_3^3)^\gamma$, $\gamma = (1, 1, -1, 1, (132))$

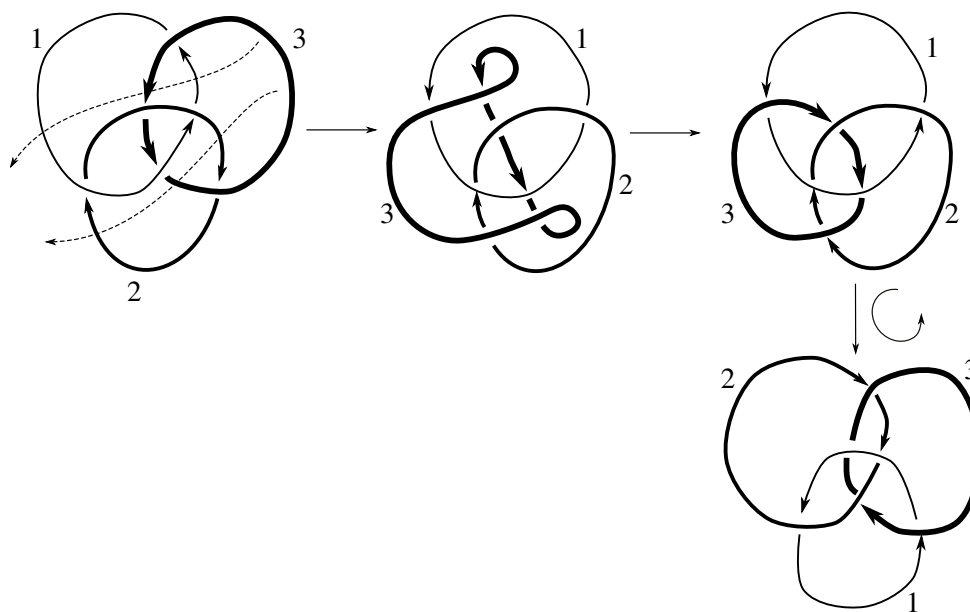


FIGURE 40. $(6_3^3)^\gamma$, $\gamma = (1, -1, -1, -1, (12))$

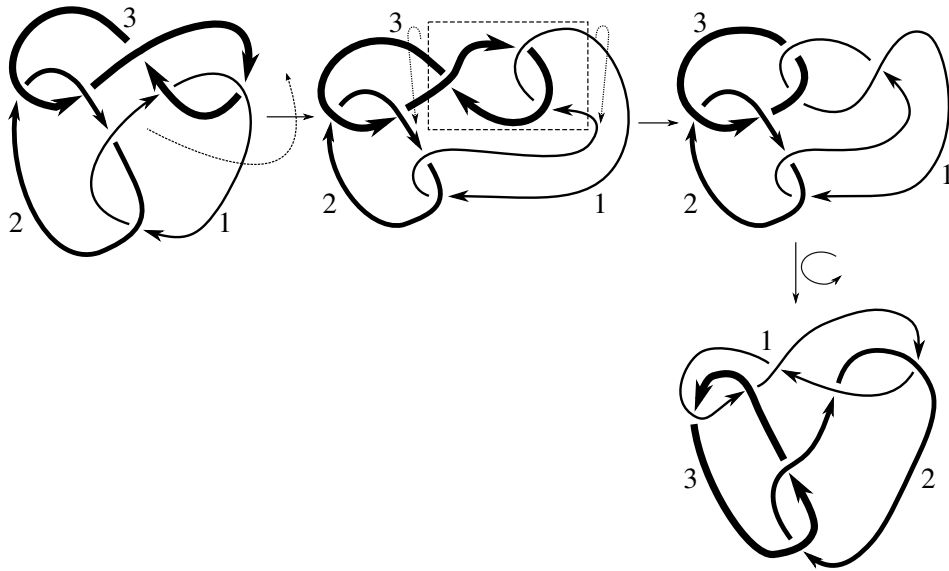


FIGURE 41. $(7_1^3)^\gamma$, $\gamma = (1, 1, -1, 1, (123))$

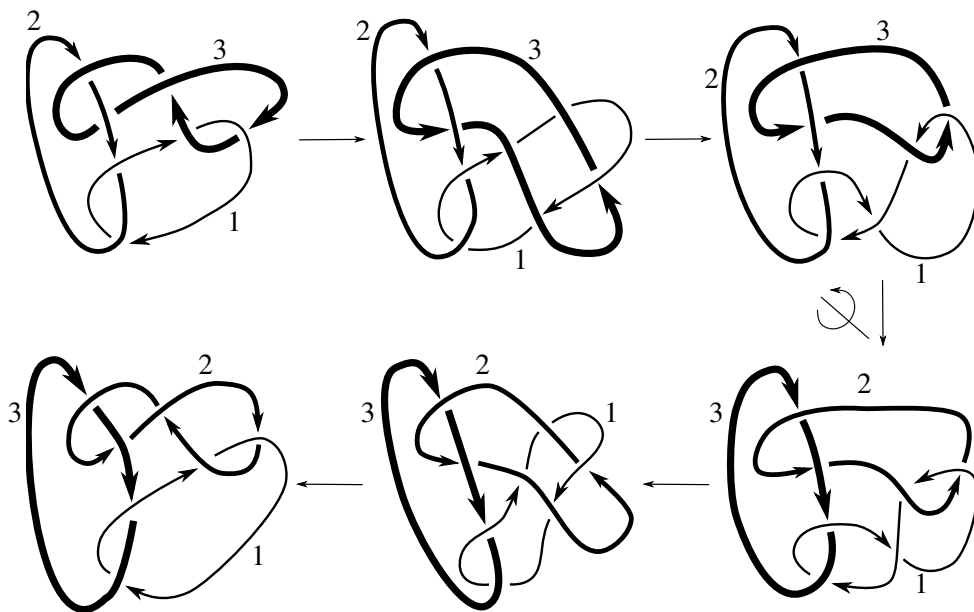


FIGURE 42. $(7_1^3)^\gamma$, $\gamma = (1, 1, 1, 1, (23))$

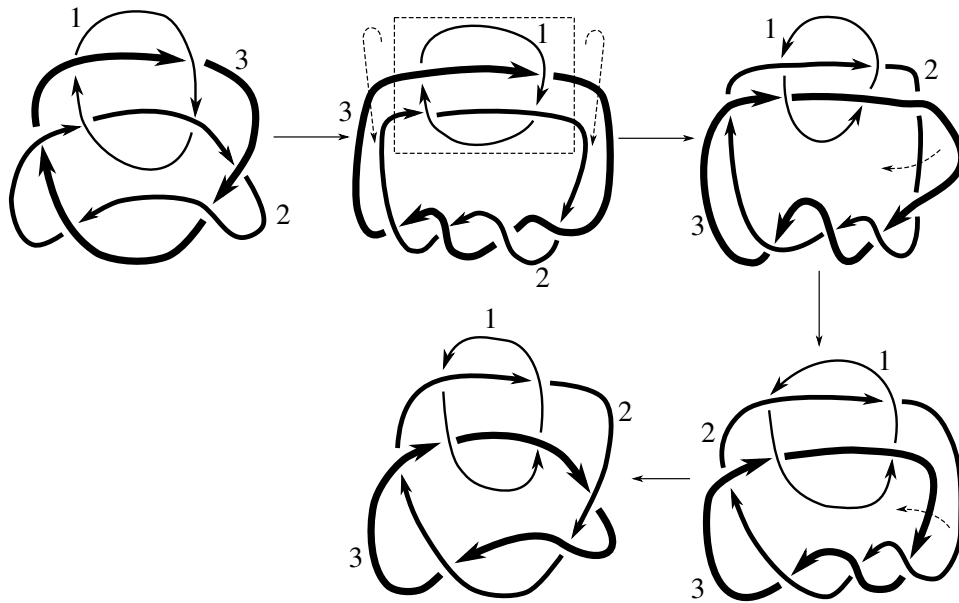


FIGURE 43. $(8_1^3)^\gamma$, $\gamma = (1, -1, 1, 1, (23))$

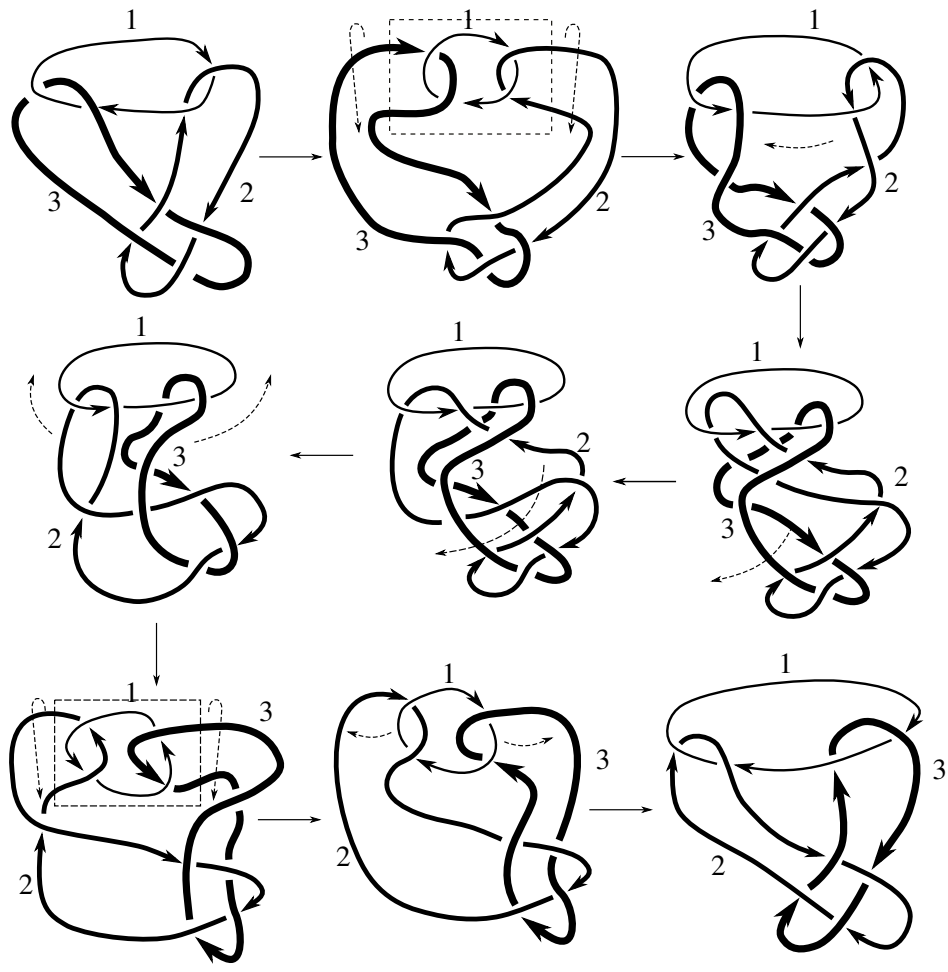


FIGURE 44. $(8_2^3)^\gamma$, $\gamma = (1, 1, 1, 1, (23))$

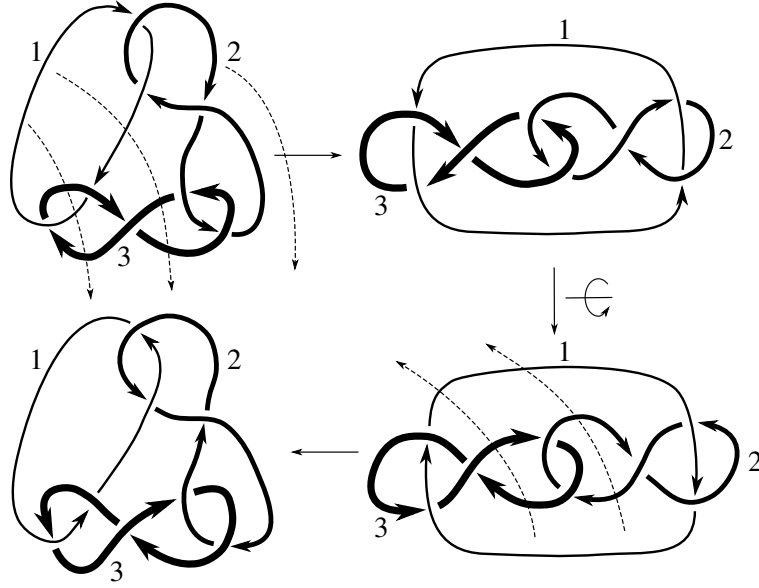


FIGURE 45. $(8_3^3)^\gamma$, $\gamma = (1, -1, -1, -1, e)$

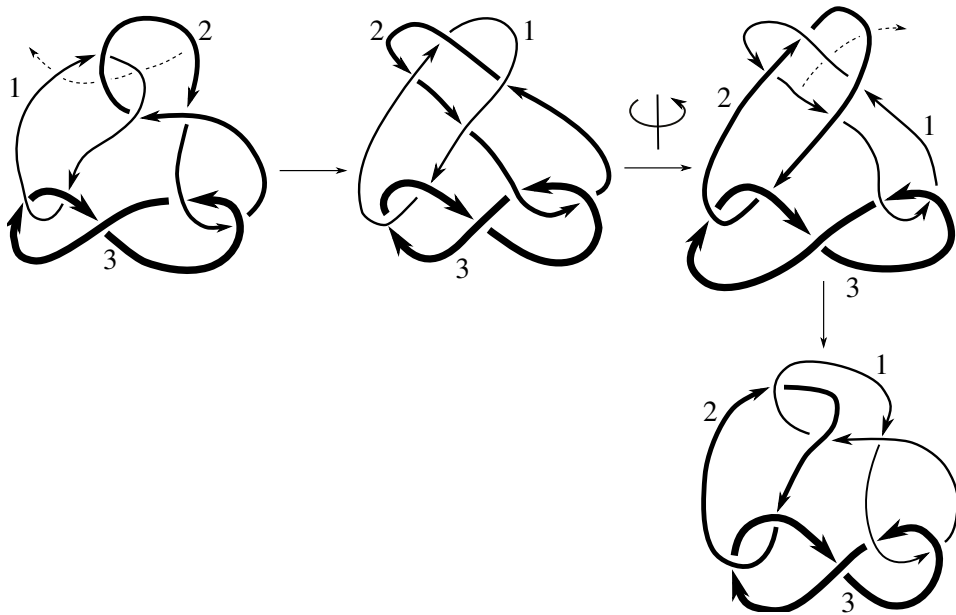


FIGURE 46. $(8_3^3)^\gamma$, $\gamma = (1, 1, 1, 1, (12))$

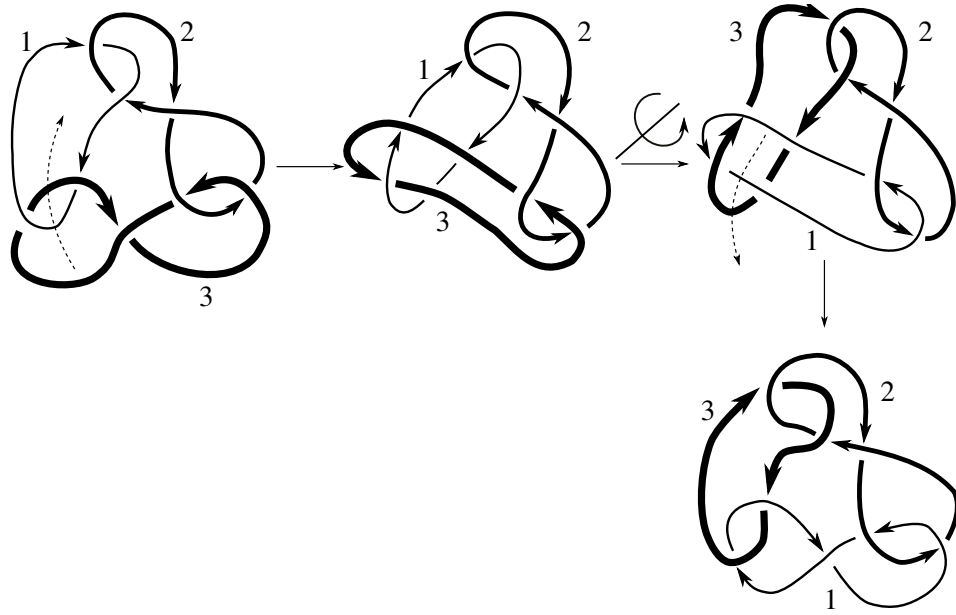


FIGURE 47. $(8_3^3)^\gamma$, $\gamma = (1, 1, 1, 1, (13))$

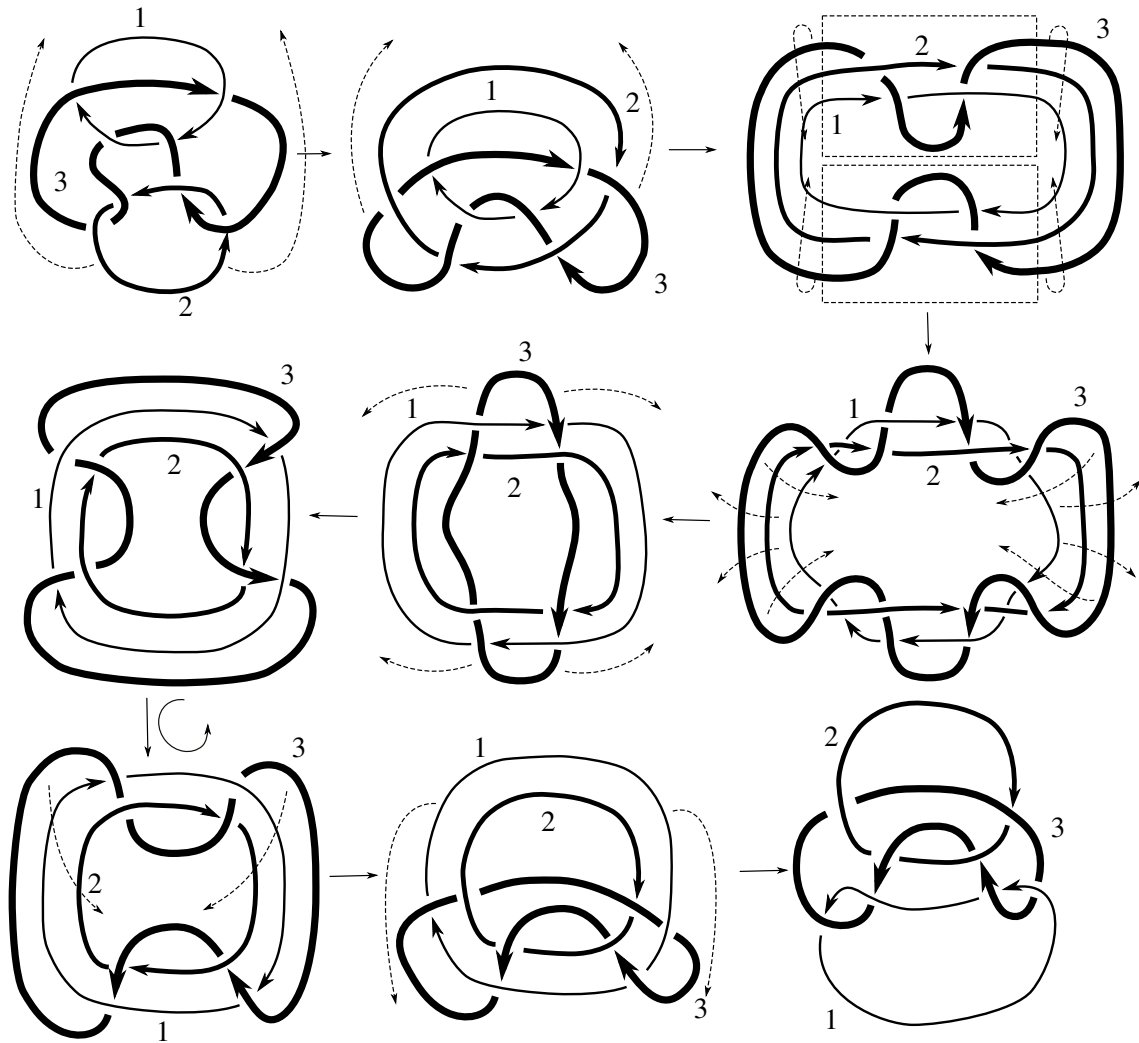


FIGURE 48. $(8_4^3)^\gamma$, $\gamma = (-1, 1, 1, 1, (12))$

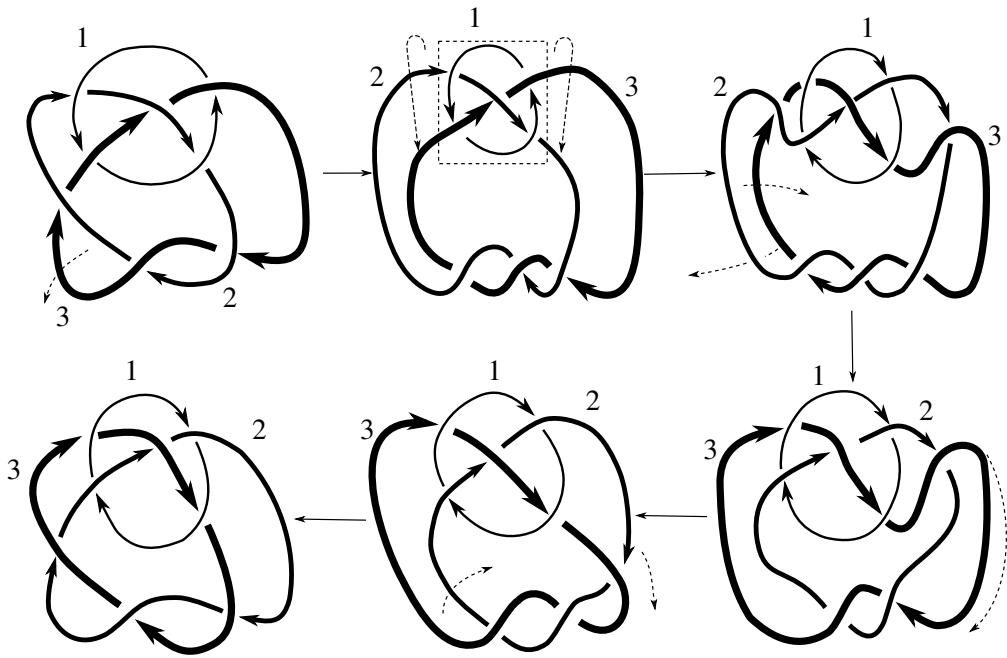


FIGURE 49. $(8_5^3)^\gamma$, $\gamma = (1, -1, 1, 1, (23))$

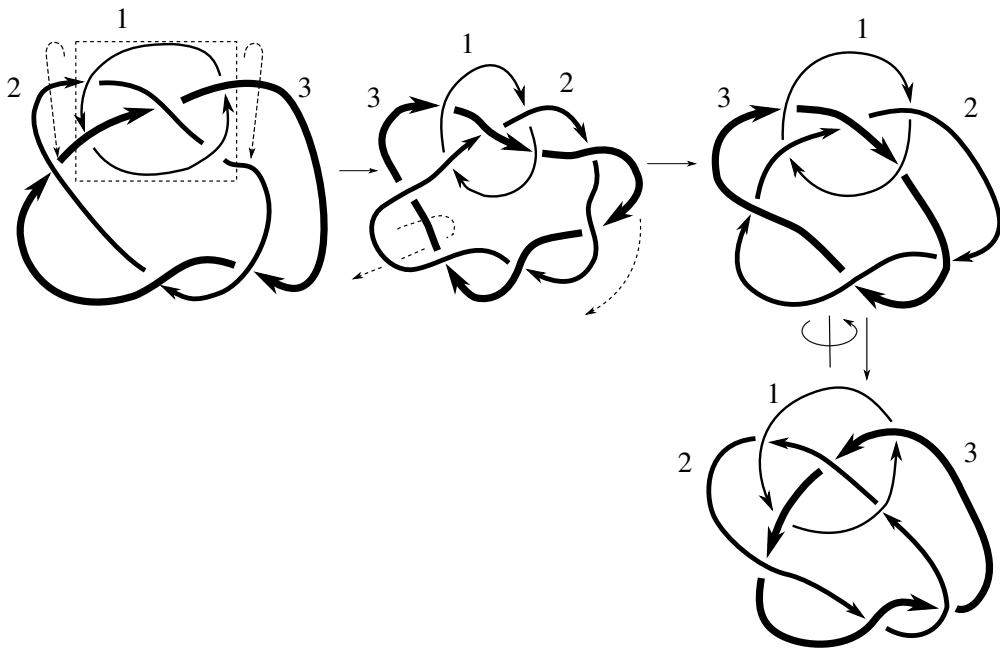


FIGURE 50. $(8_5^3)^\gamma$, $\gamma = (1, 1, -1, -1, e)$

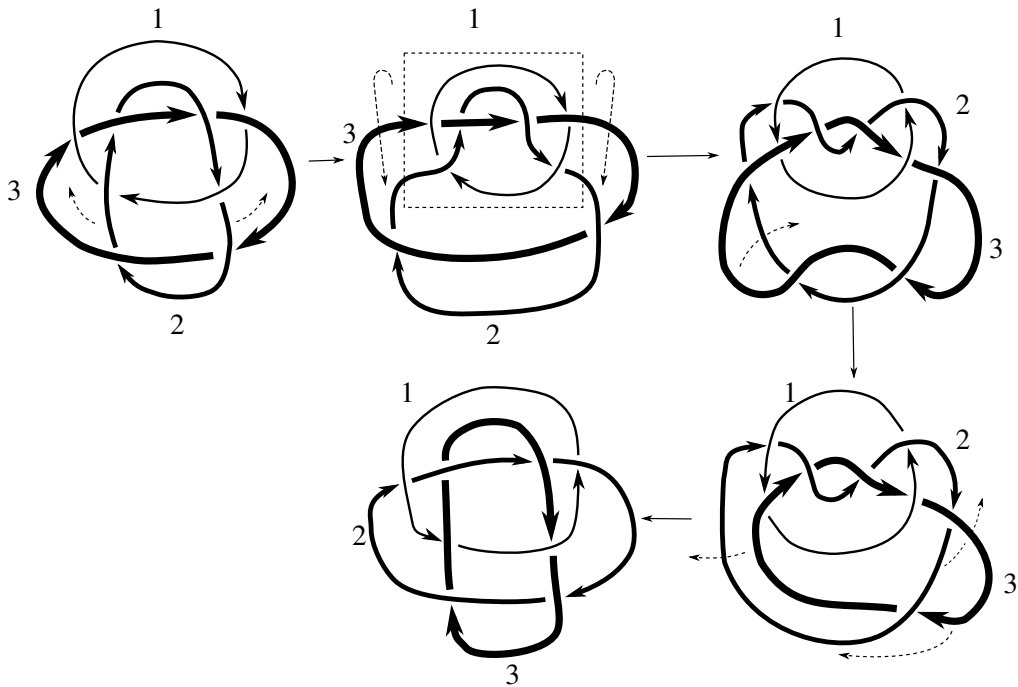


FIGURE 51. $(8_6^3)^\gamma$, $\gamma = (1, -1, 1, 1, (23))$

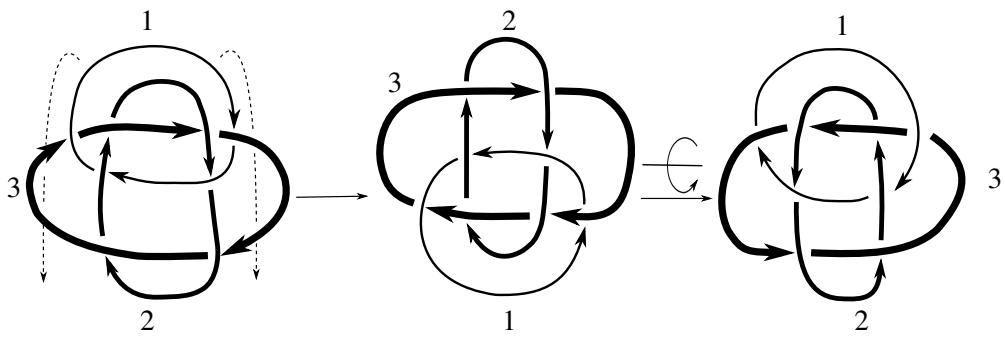


FIGURE 52. $(8_6^3)^\gamma$, $\gamma = (-1, 1, -1, -1, e)$

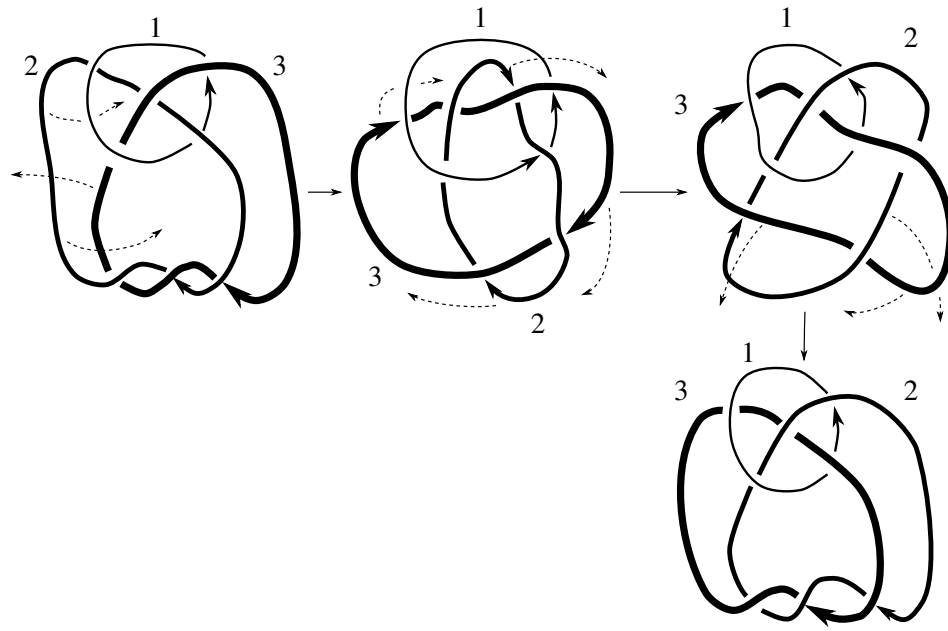


FIGURE 53. $(8_7^3)^\gamma$, $\gamma = (1, 1, 1, 1, (23))$

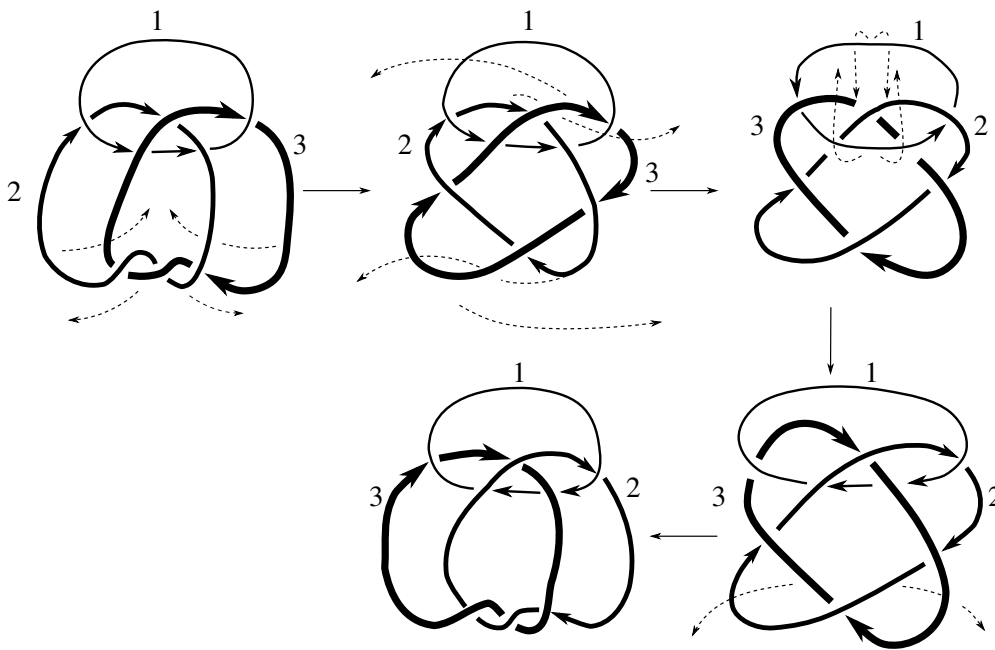


FIGURE 54. $(8_8^3)^\gamma$, $\gamma = (1, -1, 1, 1, (23))$

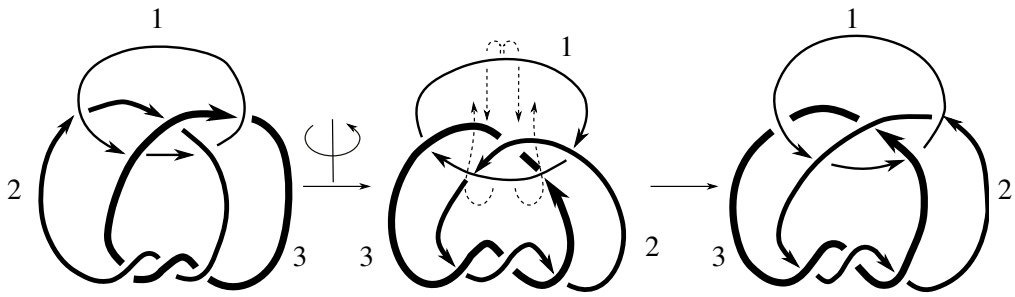


FIGURE 55. $(8_8^3)^\gamma$, $\gamma = (1, 1, -1, -1, (23))$

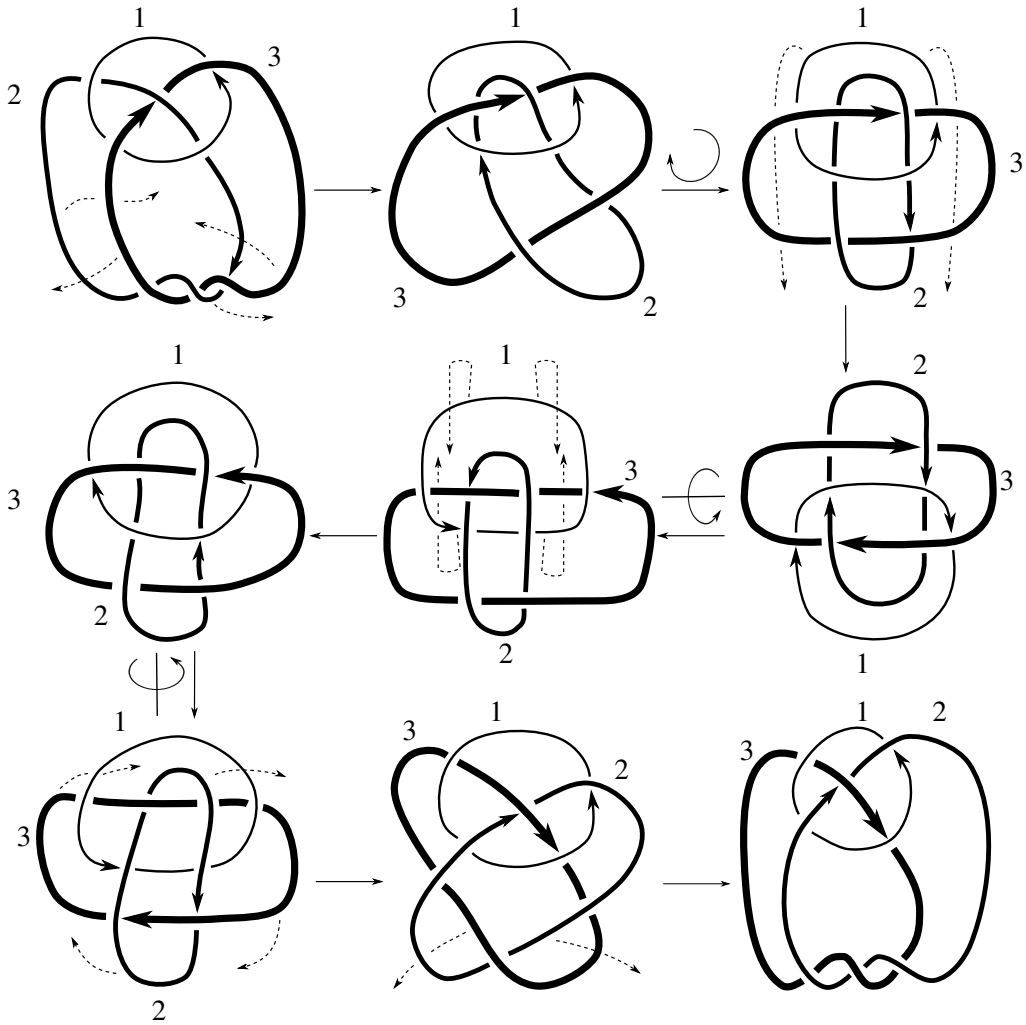


FIGURE 56. $(8_9^3)^\gamma$, $\gamma = (1, 1, 1, 1, (23))$

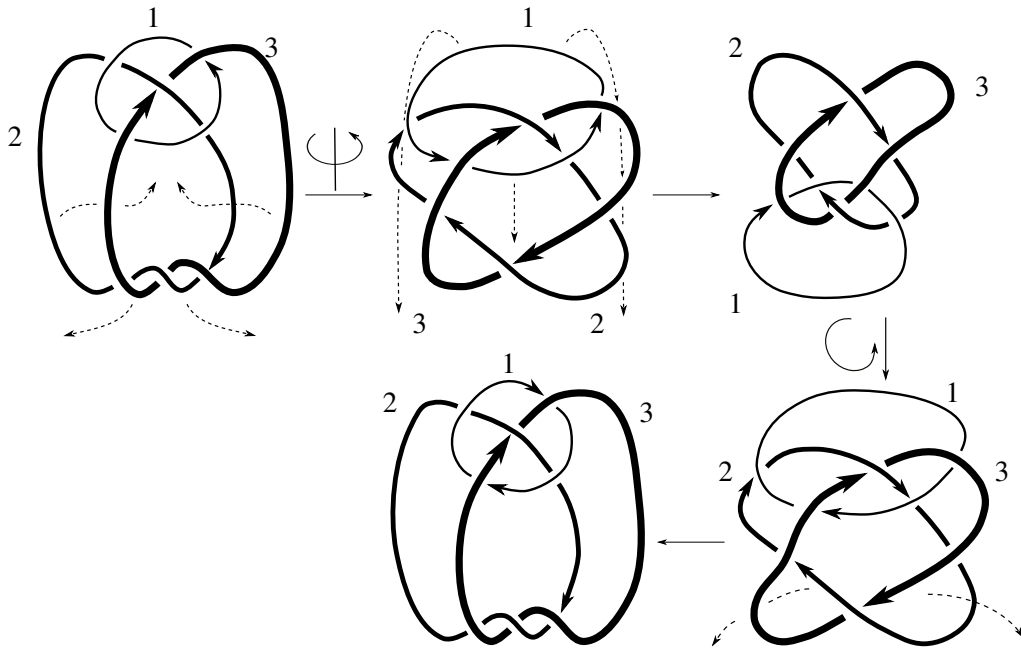


FIGURE 57. $(8_9^3)^\gamma$, $\gamma = (1, -1, 1, 1, e)$

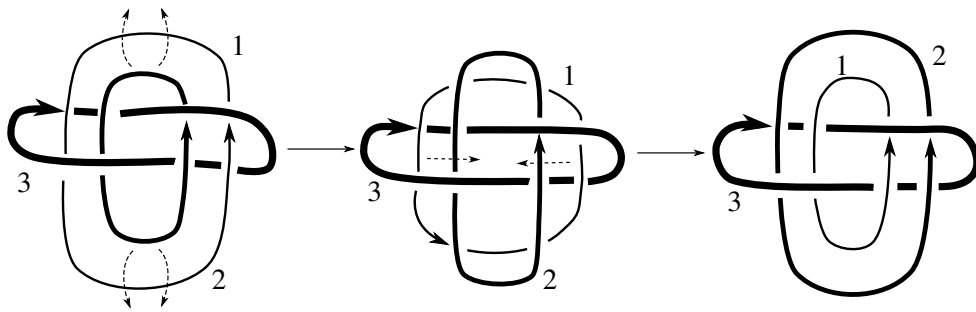


FIGURE 58. $(8_{10}^3)^\gamma$, $\gamma = (1, 1, 1, 1, (12))$

APPENDIX E. ISOTOPY FIGURES FOR FOUR-COMPONENT LINKS

This appendix contains all of the isotopy figures for three-component links that are not contained in Appendix B.

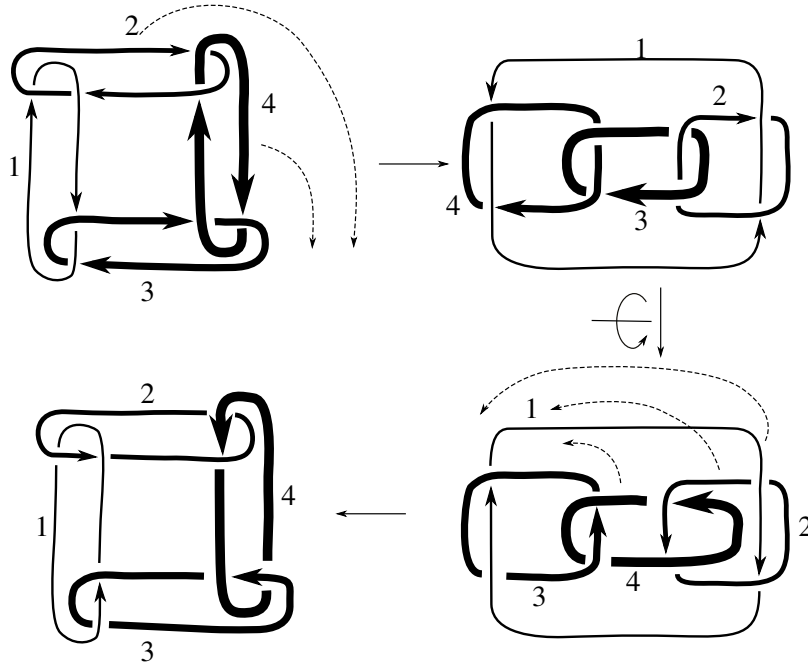


FIGURE 59. $(8_1^4)^\gamma$, $\gamma = (1, -1, -1, -1, -1, e)$

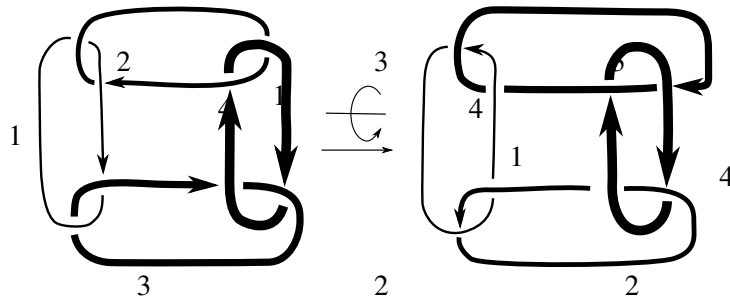


FIGURE 60. $(8_1^4)^\gamma$, $\gamma = (1, -1, 1, -1, -1, (23))$

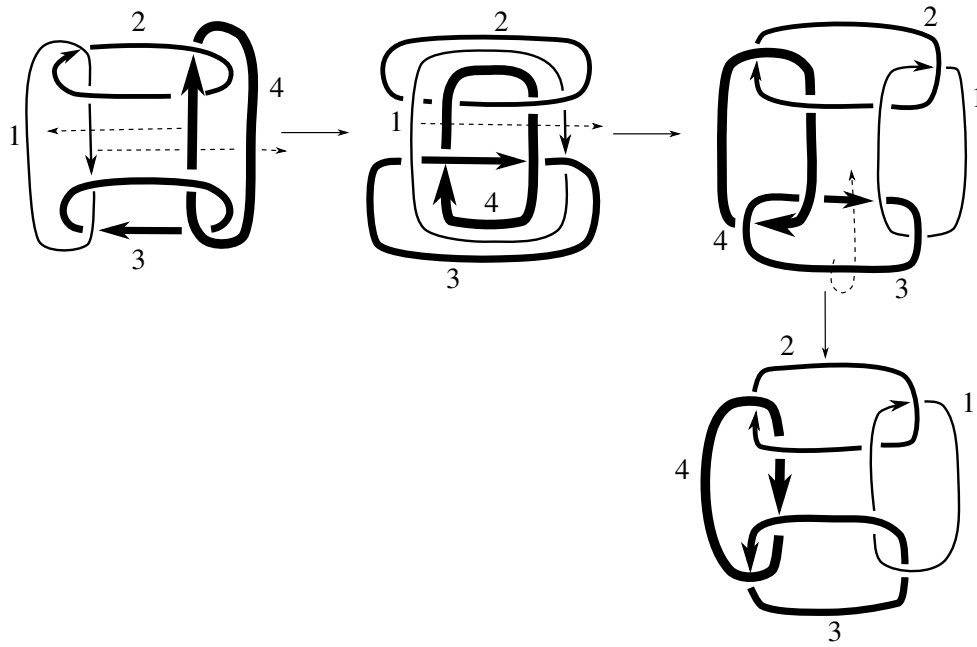


FIGURE 61. $(8_2^4)^\gamma$, $\gamma = (1, 1, 1, -1, 1, (14))$

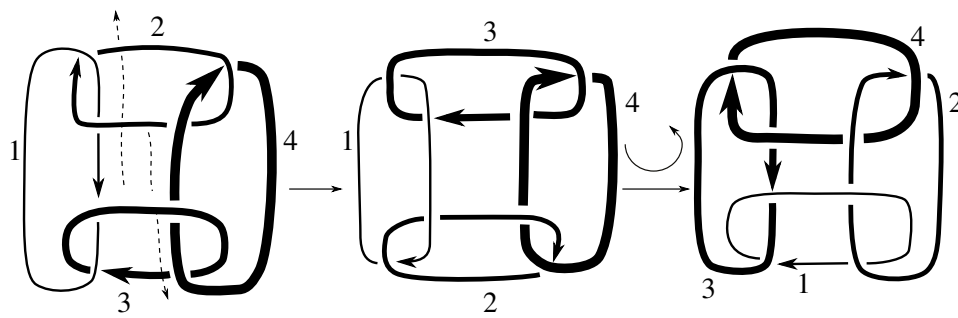


FIGURE 62. $(8_2^4)^\gamma$, $\gamma = (1, 1, 1, 1, 1, (13)(24))$

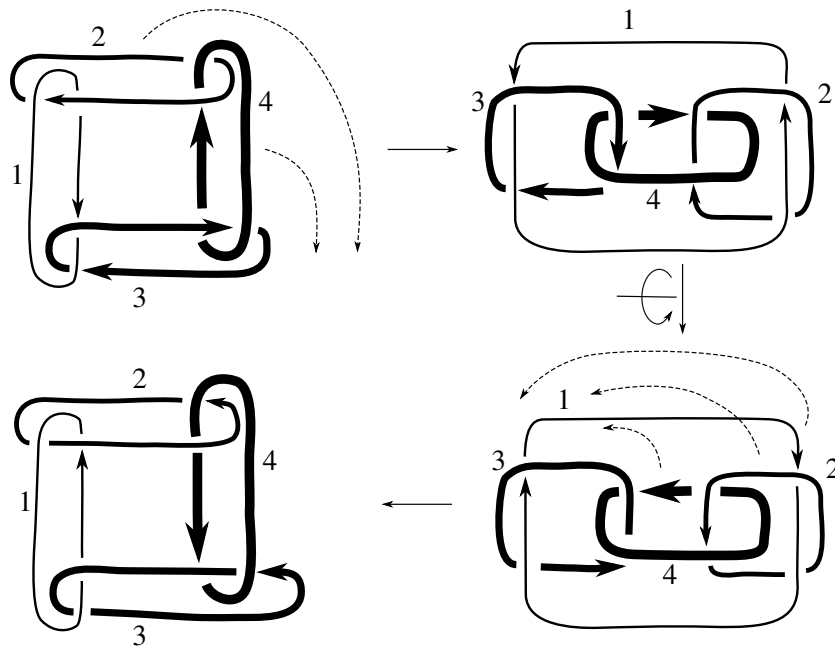


FIGURE 63. $(8_3^4)^\gamma$, $\gamma = (1, -1, -1, -1, -1, e)$

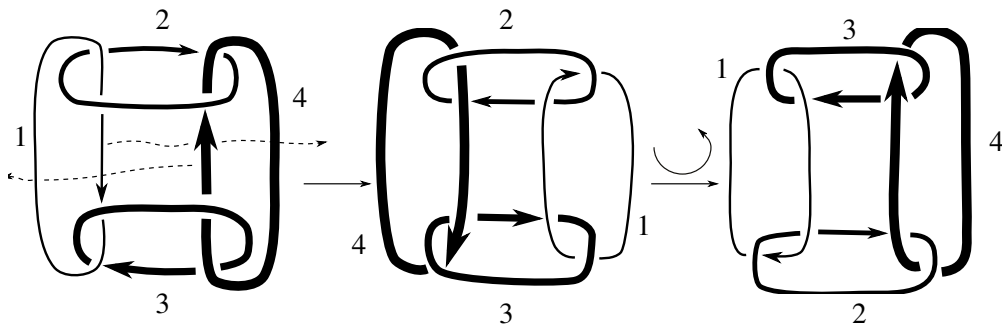


FIGURE 64. $(8_3^4)^\gamma$, $\gamma = (-1, 1, 1, 1, 1, (23))$

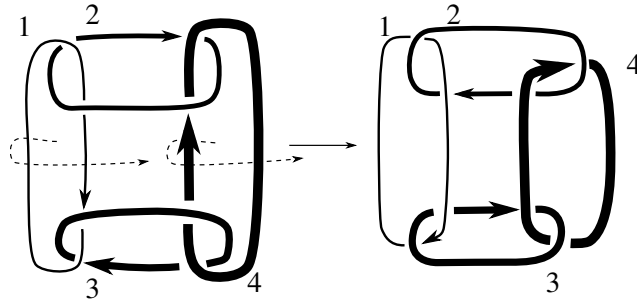


FIGURE 65. $(8_3^4)^\gamma$, $\gamma = (-1, -1, 1, 1, -1, e)$

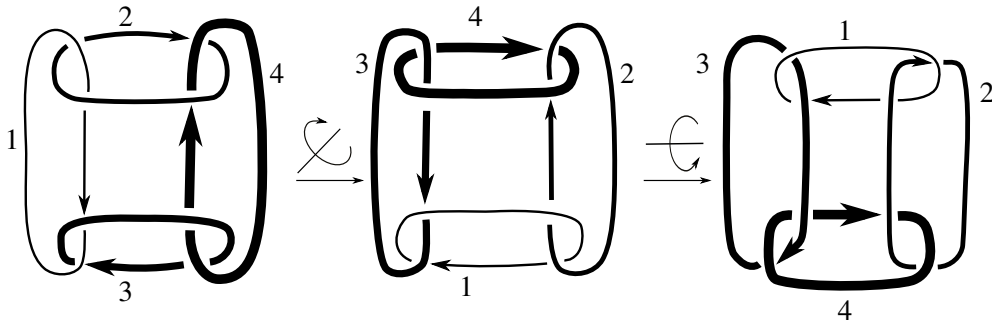


FIGURE 66. $(8_3^4)^\gamma$, $\gamma = (-1, 1, 1, 1, 1, (1342))$