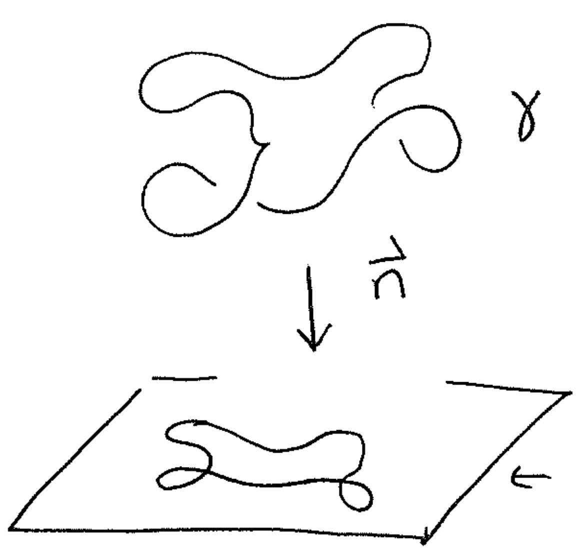
Crofton's Formula and Integralgeometric Measure.

We now want to answer some natural questions about the indicatrices. How long is the tangent indicatrix of a closed space curve? How about the normal indicatrix?

To answer these questions, we need a neat trick: averaging!



Q. What's the relationship between length of γ and length of $P_{\vec{n}}\gamma$?

Proposition. $\text{Length}(\gamma) = \frac{1}{\pi^2} \int_{\vec{n} \in S^2} \text{Length}(P_{\vec{n}}\gamma)\, d\text{Area}$

[Integralgeometric Measure]

Proof. Observe that the curve $(P_{\vec{n}}\gamma)^{\perp} = \alpha$ defined by

$$\alpha'(s) = \vec{n} \times \gamma'(s)$$

is a $\pi/2$-rotation (and maybe translation) of $P_{\vec{n}}\gamma$. Now

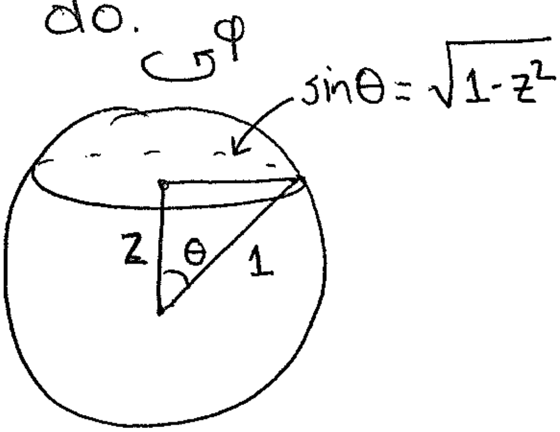
$$\int_{\vec{n} \in S^2} \text{Length}(P_{\vec{n}}\gamma)\, dA = \int_{\vec{n} \in S^2}\int_{s} \overbrace{|\vec{n} \times \vec{\gamma}'(s)|}^{|\alpha'(s)| = |P_n \gamma'|}\, ds\, d\text{Area}$$

$$= \underbrace{\int_{s}\int_{\vec{n} \in S^2}}_{\text{reversed order of integration}} \sin\theta\, |\gamma'(s)|\, d\text{Area}\, ds.$$

angle between $\vec{\gamma}'(s)$ and $\vec{n}$

$$= \int_S |\gamma'(s)| \cancel{ds} \left(\int_{\vec{n} \in S^2} \sin\theta \, dArea \right) ds$$

Now this middle integral is easy enough to do.



Fun fact: $dArea_{S^2} = dz\, d\phi$ if z is the vertical coordinate and ϕ rotates around that axis. So we most integrate

$$\int_0^{2\pi}\int_{-1}^{1} \sqrt{1-z^2}\, dz\, d\theta = 2\pi \int_{-1}^{1} \sqrt{1-z^2}\, dz$$

where ~~$\cos u$~~ $\sin u = z$, $\cos u\, du = dz$,

$$= 2\pi \int \sqrt{1-\sin^2 u}\, \cos u\, du = 2\pi \int \cos^2 u\, du$$

$$= 2\pi \int \frac{1}{2} + \frac{1}{2}\cos 2u \, du$$

$$= 2\pi \left(\frac{u}{2} + \frac{1}{4} \sin 2u \right) \Big|_{u=}$$

$$= 2\pi \left(\frac{u}{2} + \frac{1}{2}\, 2 \sin u \cos u \right) \Big|_{u=}$$

And plugging back in, $z = \sin u$, $\cos u = \sqrt{1-z^2}$, we get

$$= 2\pi \left(\frac{\arcsin z}{2} + z\sqrt{1-z^2} \right) \Big|_{z=-1}^{1}$$

$$= \pi \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = \pi^2.$$

So our original integral is

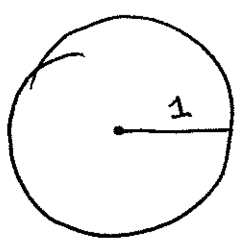
$$= \pi^2 \int |\gamma'(s)| ds = \pi^2 \, \text{Length}(\gamma). \quad \square$$

Corollary.

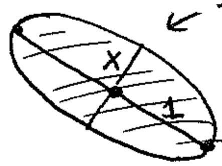
$$\text{Length}(\gamma) = \frac{4}{\pi} \left(\text{Average}_{n \in S^2} \text{Length}(P_n \gamma) \right).$$

Somehow nobody thinks of this, but it means that if you're given a collection of random pictures of a 3d curve, your guess for the length of the curve should be $\frac{4}{\pi}$ the sample average!

Example. A photograph of a coin flip likely shows ~~an~~ the coin as an ellipse with what axes?



Assume a circular coin of radius 1. From an angle, the projection is an ellipse.



← the other direction is foreshortened

some direction is in the picture plane, this has length 1

We're looking for the x where the arclength of the ellipse is $\frac{\pi}{4} \cdot (2\pi)$ or ~~the~~ $\pi^2/2$.

The arclength of the ellipse is not an integral with an elementary solution, but a little messing with Mathematica's numerical integration functions should convince you that the proper x is about 0.53.

Enough fun!

We now use a variant of this trick to compute ~~length of curves on the sphere.~~ total curvature of a projection.

Proposition. [Fáry] For any space curve $\gamma(s)$,

$$\underbrace{\int \kappa_\gamma(s)\,ds}_{\substack{\text{total curvature}\\ \text{of } \gamma}} = \frac{1}{4\pi}\int_{\vec{n}\in S^2} \underbrace{\int_s \kappa_{P_n(\gamma)}(s)\,ds}_{\substack{\text{total curvature}\\ \text{of projection}}}$$

Proof. As before, observe that just as

$$\text{Length } \alpha(s) = \text{Length } P_{\vec{n}}\gamma$$

we have

$$\int \kappa_\alpha(s)\,ds = \int \kappa_{P_{\vec{n}}(\gamma)}(s)\,ds.$$

Now observe

$$\alpha'(s) = \vec{n} \times \vec{\gamma}'(s)$$

implies

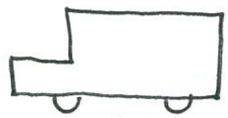
$$\alpha''(s) = \vec{n} \times \vec{\gamma}''(s)$$

So

$$\kappa_\alpha(s) = \frac{|\alpha'(s) \times \alpha''(s)|}{|\alpha'(s)|^3}$$

To analyze $\alpha'(s)\times\alpha''(s)$ we recall

$$a\times(b\times c) = \underset{\substack{\uparrow\\ \text{cab}}}{\langle c,a\rangle b} - \underset{\substack{\uparrow\\ \text{back!}}}{\langle b,a\rangle c}$$



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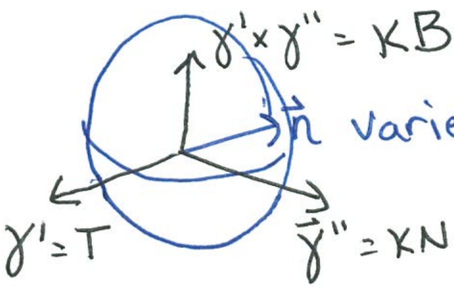
So

$$\alpha'(s)\times(\vec{n}\times\vec{\gamma}''(s)) = \langle\alpha',\gamma''\rangle\vec{n} - \underbrace{\langle\vec{n},\vec{\alpha}'\rangle}_{\text{zero!}}\gamma''.$$

and

$$\begin{aligned}\langle\alpha',\gamma''\rangle &= \langle \vec{n}\times\vec{\gamma}',\vec{\gamma}''\rangle\\ &= \langle\vec{\gamma},\vec{n}\times\vec{\gamma}'\rangle\\ &= \langle\vec{n},\vec{\gamma}'\times\vec{\gamma}''\rangle = \cos\theta_B\,|\vec{\gamma}'\times\vec{\gamma}''| = \kappa\cos\theta_B.\end{aligned}$$

So we have the situation



We have now shown that

$$\int_{\vec{n}\in S^2}\int_S \kappa_{P_n(\gamma)}(s)\,ds = \int_S \kappa(s)\left(\int_{\vec{n}\in S^2} \frac{\cos\theta_B}{(\sin\theta_T)^3}\,dArea\right) ds.$$

where θ_B is the angle between $\vec{n}$ and B and θ_T is the angle between $\vec{n}$ and T.

Option 1. We can choose coordinates so that $T=(1,0,0)$ and $B=(0,1,0)$ and integrate in coordinates.

Option 2. We can observe that this is independent of the direction of T, B and so it's a constant λ, so

$$\int \kappa_\gamma(s)\,ds = \lambda\cdot\int_{\vec{n}\in S}\int_S \kappa_{P_n(\gamma)}(s)\,ds$$

If γ = circle, the total curvature of any projection is 2π, so $2\pi = \lambda\cdot 4\pi\cdot 2\pi$

and we have learned $\lambda = 1/4\pi$. $\square$

Note. We still haven't checked that total curvature of the ellipse $= 2\pi$, but it follows from the fact that the tangent indicatrix is the equator.

The examples are probably obvious; and again give you a way to calculate or estimate curvature from photographs of a 3d object.

Research project. Projections along X, Y, Z axes give lengths and total curvatures L_x, L_y, L_z and K_x, K_y, K_z. Find bounds on L and K in terms of these numbers.