More on symmetry and majorization.

Suppose

$$(x+a_1)\cdots(x+a_n) = x^n + c_1 x^{n-1} + \dots + c_n$$

$$= x^n + \binom{n}{1} p_1 x^{n-1} + \binom{n}{2} p_2 x^{n-2} + \dots + p_n$$

The c_i are the elementary symmetric functions of the a_i – that is

$$c_1 = a_1 + \dots + a_n$$

$$c_2 = a_1 a_2 + a_1 a_3 + \dots = \sum_{i<j} a_i a_j$$

$$c_k = \sum_{i_1 < \dots < i_k} a_{i_1} \cdots a_{i_k}$$

$$c_n = a_1 \cdots a_n$$

while the p_i are the corresponding averages (instead of sums).

All of the p_i are symmetric means of the a_i, since

$$\& \quad C_r = \underbrace{\left(\sum{}^! a_1 \cdots a_r\right)}_{\text{each of these occurs multiple times}} \cdot \frac{1}{r!\,(n-r)!}$$

ways to rearrange n-r terms left out of product

ways to rearrange r terms in product

and

$$p_r = \frac{1}{\binom{n}{r}} C_r = \frac{1}{\frac{n!}{r!\,(n-r)!}} C_r = \frac{1}{n!} \sum{}^! a_1 \cdots a_r = [\underbrace{1,1,\ldots,1}_{r \text{ times}},0,\ldots,0]$$

Now if $p_r = [\alpha]$, then $\sum \alpha_i = r$, so different p_r can't majorize one another and so can't be directly comparable. They can (and do!) obey non linear inequalities.

Theorem (Newton) $p_{r-1}\,p_{r+1} \leq p_r^2$, for any real nonzero a_i (including negative ones!), with equality only if all a_i are equal.

$$= \frac{n!}{(n-(i+j))!} \frac{(n-(i+j))!}{(n-(i+j)-(k-j))!\,(k-j)!} P_K x^{n-k-i} y^{k-j}$$

$$= \frac{n!}{(n-(i+j))!} \binom{n-(i+j)}{k-j} P_K x^{n-k-i} y^{k-j}$$

Now ~~[illegible]~~ let's pick some r and let $i = n-r-1$, $j = r-1$, so $i+j = n-2$. Then most of the x^{n-k-i} and y^{k-j} vanish. Simplifying, the general term is

$$= \frac{n!}{2!} \binom{2}{k-r+1} P_K x^{r+1-k} y^{k-r+1}$$

and this is nonzero only when

$$k-r+1 = 2, \qquad k-r+1 = 1, \qquad k-r+1 = 0$$

Lemma. If

$$f(x,y) = c_0 x^m + c_1 x^{m-1} y + \dots + c_m y^m = 0$$

has all roots x/y real, then same is true of all roots of

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y) = 0. \quad \left(\text{unless } \frac{\partial^{i+j}}{\partial x^i \partial y^j} f(x,y) \text{ vanishes identically.}\right)$$

Further, if α is a root of multiplicity $M>1$ of one of these equations E, it is a root of multiplicity $M+1$ of the equation ~~from~~ E' ~~which~~ for which $\frac{\partial}{\partial x} E' = E$ or $\frac{\partial}{\partial y} E' = E$.

Proof. Exercise.

Suppose that

$$f(x,y) = (x + a_1 y) \cdots (x + a_n y)$$

$$= p_0 x^n + \binom{n}{1} p_1 x^{n-1} y + \dots + \binom{n}{n} p_n x^0 y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} p_k x^{n-k} y^k$$

We compute

$$\frac{\partial^{i+j}}{\partial x^i \partial y^j} \binom{n}{k} p_k x^{n-k} y^k =$$

$$= \binom{n}{k} p_k (n-k)(n-k-1)\cdots(n-k-i+1)\, k(k-1)\cdots(k-j+1) \times$$

$$\times\, x^{n-k-i} y^{k-j}$$

$$= \binom{n}{k} p_k \frac{(n-k)!}{(n-k-i)!} \frac{k!}{(k-j)!} x^{n-k-i} y^{k-j}$$

$$= \frac{n!}{\cancel{k!}\cancel{(n-k)!}} \frac{\cancel{(n-k)!}}{(n-k-i)!} \frac{\cancel{k!}}{(k-j)!} p_k x^{n-k-i} y^{k-j}$$