# Hölder's ~~and~~ ~~[illegible]~~ Theorem~~s~~

We have now defined

$$M_r(\vec{a}) = \left( \frac{\sum p_i a_i^r}{\sum p_i} \right)^{1/r}, \quad M_0(\vec{a}) = \left(a_1^{p_1} \cdots a_n^{p_n}\right)^{1/\sum p_i}$$

and proved that for $r > 0$, $M_r(\vec{a}) \leq M_{2r}(\vec{a})$.

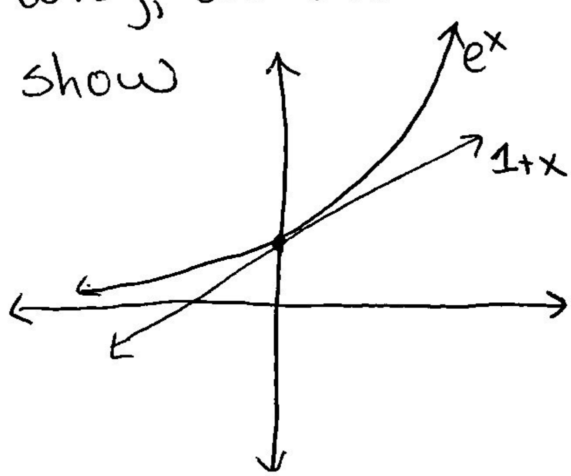
Proposition. $M_1(\vec{a}) = A(\vec{a}) \geq G(\vec{a}) = M_0(\vec{a})$ with equality only if all $a_i$ are equal.

Proof 1.

$$M_1(\vec{a}) \geq M_{1/2}(\vec{a}) \geq M_{1/4}(\vec{a}) \geq \cdots \geq \lim_{n\to\infty} M_{1/2^n}(\vec{a}) = G(\vec{a})$$

Proof 2. (Pólya)

Wlog, we assume $\sum p_i = 1$. Now we can show




$1 + x \leq e^x$ (equality $x=0$)

$x \leq e^{x-1}$ (equality, $x=1$)

We now define

$$\alpha_K = \frac{a_K}{U(\vec{a})} = \frac{a_K}{p_1 a_1 + \cdots + p_n a_n}$$

Now

$$\alpha_K \le e^{\alpha_K - 1}, \quad \text{so} \quad \alpha_K^{p_K} \le e^{p_K \alpha_K - p_K}.$$

This means

$$\alpha_1^{p_1} \cdots \alpha_n^{p_n} \le e^{(\sum p_K \alpha_K) - 1} = e^{1-1} = 1.$$

But

$$\alpha_1^{p_1} \cdots \alpha_n^{p_n} = \frac{a_1^{p_1} \cdots a_n^{p_n}}{U^{p_1} \cdots U^{p_n}} = \frac{G(\vec{a})}{U(\vec{a})}. \quad \square$$

Cute, right? Steele has a nice disussion of how Pólya may have come up with this idea.

We now push forward.

Proposition. (Hölder Inequality)
Suppose $\vec{a}_1, \ldots, \vec{a}_m$ are non-negative vectors in $\mathbb{R}^n$ (as usual). Then

$$G(\vec{a}_1) + \ldots + G(\vec{a}_m) \leq G(\vec{a}_1 + \ldots + \vec{a}_m)$$

with equality only if (all the $\vec{a}_i$ are linearly dependent <u>or</u> there is some $j$ so that $a_{1j} = a_{2j} = \ldots = a_{mj} = 0$.).

Before we prove this, it will be helpful to rewrite it. The inequality says (assuming as usual $\sum p_i = 1$),

$$(a_{11}^{p_1} \cdots a_{1n}^{p_n}) + (a_{21}^{p_1} + \ldots + a_{2n}^{p_n}) + \ldots + (a_{m1}^{p_1} \cdots a_{mn}^{p_n})$$

$$\leq (a_{11} + a_{21} + \ldots + a_{m1})^{p_1} \cdots (a_{1n} + \ldots + a_{mn})^{p_n}$$

If we think about the $\vec{a}_{ij}$ as a matrix, it's ok to take the transpose and rewrite the theorem as

Suppose $\vec{a}_1, \dots, \vec{a}_n$ are non-negative vectors in $\mathbb{R}^m$, then

$$\sum_i a_{1i}^{p_1} \cdots a_{ni}^{p_n} \leq \left(\sum_i a_{1i}\right)^{p_1} \cdots \left(\sum_i a_{ni}\right)^{p_n}$$

with equality only if (all $\vec{a}_i$ are linearly dependent <u>or</u> some $\vec{a}_i = (0, \dots, 0)$).

Proof. We have (assuming <u>no</u> $\vec{a}_i = \vec{0}$)

$$\frac{\sum_i a_{1i}^{p_1} \cdots a_{ni}^{p_n}}{\left(\sum_j a_{1j}\right)^{p_1} \cdots \left(\sum_j a_{nj}\right)^{p_n}} = \sum_{i=1}^{m} \left(\frac{a_{1i}}{\sum_j a_{1j}}\right)^{p_1} \cdots \left(\frac{a_{ni}}{\sum_j a_{nj}}\right)^{p_n}$$

Now apply the theorem $U(\vec{a}) \geq G(\vec{a})$ $m$ times (that is, <u>inside</u> the sum over $i$).

we get

$$\leq \sum_i \left( p_1 \frac{a_{1i}}{\sum_j a_{1j}} + \dots + p_n \frac{a_{ni}}{\sum_j a_{nj}} \right)$$

$$= p_1 + \dots + p_n = \underline{1}.$$

We get equality only if <u>all</u> applications of $G(\vec{a}) \leq U(\vec{a})$ were equalities, or if (for all i),

$$\frac{a_{1i}}{\sum_j a_{1j}} = \frac{a_{2i}}{\sum_j a_{2j}} = \dots = \frac{a_{ni}}{\sum_j a_{nj}}$$

or

$$\frac{\vec{a}_1}{\sum_j a_{1j}} = \dots = \frac{\vec{a}_n}{\sum_j a_{nj}},$$ which is the same as "all $\vec{a}_i$ are linearly dependent".

If some $\vec{a}_i = \vec{0}$, the inequality reduces to $0 = 0$. $\square$

We have expressed this as a result about the <u>sum</u> of geometric means. But a clever substitution rewrites this as a relation about the <u>product</u> of r-means.

Proposition. If $r, p_1, \ldots, p_m$ are positive, and $\vec{a}_1, \ldots, \vec{a}_m$ are nonnegative in $\mathbb{R}^n$, (and $\sum p_i = 1$) then

$$M_r(\vec{a}_1 \cdot \ldots \cdot \vec{a}_m) \leq M_{r/p_1}(\vec{a}_1) \cdot \ldots \cdot M_{r/p_m}(\vec{a}_m)$$

with equality only if ( $\vec{a}_1^{\,1/p_1}, \ldots, \vec{a}_m^{\,1/p_m}$ are linearly dependent or some $\vec{a}_i = \vec{0}$ ).

Proof. Rewriting Hölder,

$$\sum b_{1i}^{p_1} \cdots b_{mi}^{p_m} \leq \Big(\sum_i b_{1i}\Big)^{p_1} \cdots \Big(\sum_i b_{mi}\Big)^{p_m}$$

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Let $q_1, \dots, q_n$ be the weights for the $M_r$ norms in the statement, and let

$$\vec{b}_{ij} = q_j \vec{a}_{ij}^{\,r/p_i}$$

Then the lhs becomes

$$\sum_{i=1}^{n} q_i^{p_1} \vec{a}_{1i}^{\,r/} \cdots q_i^{p_m} \vec{a}_{mi}^{\,r/}$$

$$= \sum_{i=1}^{n} q_i \, \vec{a}_{1i}^{\,r} \cdots \vec{a}_{mi}^{\,r}$$

$$= M_r(\vec{a}_1 \cdots \vec{a}_m)^r.$$

The rhs becomes

$$\left(\sum_{i=1}^{n} q_i \vec{a}_{1i}^{\,r/p_1}\right)^{p_1} \cdots \left(\sum_{i=1}^{n} q_i \vec{a}_{mi}^{\,r/p_m}\right)^{p_m}$$

$$= M_{r/p_1}(\vec{a}_1)^r \cdots M_{r/p_m}(\vec{a}_m)^r \qquad \square$$

~~Proof~~.

When there are only two vectors $\vec{a}, \vec{b}$, we can specialize as follows.

Definition. If $(k-1)(k'-1) = 1$, we say $k, k'$ are conjugate. If neither is $0$, this is equivalent to $\frac{1}{k} + \frac{1}{k'} = 1$.

Theorem. If $k, k'$ are conjugate, then

$$\sum a_i b_i \leq \left(\sum a_i^k\right)^{1/k} \left(\sum b_i^{k'}\right)^{1/k'} \quad (k > 1)$$

(as usual all $a_i, b_i \geq 0$).

$$\sum a_i b_i \geq \left(\sum a_i^k\right)^{1/k} \left(\sum b_i^{k'}\right)^{1/k'} \quad (k < 1).$$

with equality only if $\vec{a}^k, \vec{b}^{k'}$ are linearly dependent or $\vec{a}\vec{b} = \vec{0}$.

Proof. We return to

$$M_r(\vec{a}_1\vec{a}_2) \le M_{r/p_1}(\vec{a}_1) M_{r/p_2}(\vec{a}_2)$$

and substitute $r/p_1 = K$, $r/p_2 = K'$ and $r = 1$. Since $K, K'$ are conjugate,

$$p_1 + p_2 = \frac{1}{K} + \frac{1}{K'} = 1$$

as required, and since $K > 1$, $K' > 0$, so $p_1$ and $p_2$ are positive, as required. We get

$$M_1(\vec{a}\vec{b}) \le M_K(\vec{a}) M_{K'}(\vec{b}).$$

which is the first part of the claim. ~~We note (but don't prove) that the inequality reverses when $K < 1$.~~

To prove the other case, we make appropriate substitutions, but I don't think we learn enough to do the algebra now (p. 25 of HLP if you're curious). □

Proposition. (Hölder, very symmetric form) If $K$ and $K'$ are conjugate ($K \neq 0,1$)

$$\left(\sum a_i b_i\right)^{KK'} \leq \left(\sum a_i^K\right)^{K'} \left(\sum b_i^{K'}\right)^K$$

(with equality only when $\vec{a}, \vec{b}$ linearly dependent).
This is the form most easily reduced to Cauchy's inequality - just observe that 2 is conjugate to itself to get

$$\left(\sum a_i b_i\right)^4 \leq \left(\sum a_i^2\right)^2 \left(\sum b_i^2\right)^2$$

and take the 4th root of both sides.

We can now prove the theorem of the means!

Theorem. If $r<s$ then $M_r(\vec{a}) \leq M_s(\vec{a})$ with equality only if (all $a_i$ are equal or ($s \leq 0$ and some $a_i = 0$)).

Proof. Suppose $0<r<s$, and write $r = s\alpha$ where $0<\alpha<1$. Define $\vec{u}, \vec{v}$ by

$$u_i = p_i a_i^s \quad \text{and} \quad v_i = p_i$$

Now

$$p_i a_i^r = p_i a_i^{s\alpha} = (p_i a_i^s)^\alpha p_i^{1-\alpha} = u_i^\alpha v_i^{1-\alpha}$$

By Hölder,

$$\sum u_i^\alpha v_i^{1-\alpha} \leq \left(\sum u_i\right)^\alpha \left(\sum v_i\right)^{1-\alpha}$$

with equality only if $\vec{u}, \vec{v}$ are linearly dependent or one of $\vec{u}, \vec{v}$ is $\vec{0}$. This happens only when all $a_i$ are equal, as the weights $p_i > 0$ (and if all $a_i = 0$, they are equal!).

We have carefully engineered things so Hölder says

$$\sum p_i a_i^r \leq \left(\sum p_i a_i^s\right)^{\alpha} \left(\sum p_i\right)^{1-\alpha}$$

We now raise each side to the $1/r = 1/s\alpha$.

$$\left(\sum p_i a_i^r\right)^{1/r} \leq \left(\sum p_i a_i^s\right)^{\alpha/s\alpha} \left(\sum p_i\right)^{1/s\alpha - \alpha/s\alpha}$$

dividing by $\left(\sum p_i\right)^{1/s\alpha} = \left(\sum p_i\right)^{1/r}$,

$$\left(\frac{\sum p_i a_i^r}{\sum p_i}\right)^{1/r} \leq \left(\frac{\sum p_i a_i^s}{\sum p_i}\right)^{1/s}$$

which is $M_r(\vec{a}) \leq M_s(\vec{a})$.

We now have a host of other little cases to dispose of.

If $r \le 0$ and some $a_i = 0$, then $M_r(\vec{a}) = 0$. This is always $\le M_s(\vec{a})$ and equality occurs only when $M_s(\vec{a}) = 0$ (that is, $s \le 0$, given that we assumed some $a_i = 0$).

We are left with cases where

$r \le 0$ and all $a_i > 0$.

Suppose $r = 0$. Then if $s > 0$,

$$M_0(\vec{a})^s = G(\vec{a})^s = G(\vec{a}^s) < U(\vec{a}^s) = M_s(\vec{a})^s.$$

If $s = 0$ and $r < s$, then

$$M_r(\vec{a}) = \underbrace{\frac{1}{M_{-r}(1/\vec{a})} < \frac{1}{M_0(1/\vec{a})}}_{} = M_0(\vec{a})$$

$-r > 0$, so the previous $\Rightarrow$

$M_{-r}(1/\vec{a}) > M_0(1/\vec{a})$

If $r < 0 < s$, these two give us

$$M_r(\vec{a}) < M_0(\vec{a}) < M_s(\vec{a})$$

The final case, $r < s < 0$ is easy

$$M_r(\vec{a}) = \underbrace{\frac{1}{M_{-r}(1/\vec{a})} < \frac{1}{M_{-s}(1/\vec{a})}}_{\substack{-r > -s > 0, \text{ so} \\ M_{-r}(1/\vec{a}) > M_{-s}(1/\vec{a})}} = M_s(\vec{a})$$

And that completes our work for the day! $\square$