Holonomy and Gauss-Bonnet (II).

We have now learned that

$$\int_{\partial R} \varphi_{12}(s)\, ds = -\iint_R K \, dArea$$

Now. we Know that if

$$\alpha'(s) = \cos\theta \, e_1 + \sin\theta \, e_2$$

(remember - <u>everything</u> on the rhs depends on s) that

$$K_g(s) = \varphi_{12} + \theta'$$

so for a closed curve $\alpha(s)$ bounding a simply connected region R in the uv plane

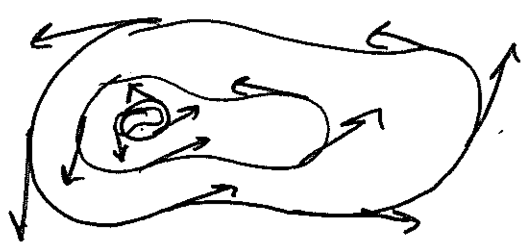
$$\int_{\partial R} K_g(s) = \int_{\partial R} \theta'(s)\, ds - \iint_R K \, dArea.$$

Now we can ~~defo~~ see that if the

curve has length L, then $\alpha'(0) = \alpha'(L)$

$$\int_0^L \theta'(s)\,ds = 2\pi K, \text{ for } K \in \mathbb{Z}.$$

But what is K? Well, if we let α shrink to a point continuously then we can see that



we're measuring the winding number (in the u-v plane), which is $+1$ (once around the circle), so $K = 1$.

Thus we have:

Theorem. If R is parametrized by a simply connected region in the uv plane, with ∂R smooth

$$\int_{\partial R} K_g\,ds + \iint_R K\,dA = 2\pi.$$

Proof. Approximate the curve by smooth curves and note that the total geodesic curvature of a rounded off corner converges to the exterior angle. □

Corollary. For a geodesic triangle with <u>interior</u> angles $\varphi_1, \varphi_2, \varphi_3$,

$$\iint_R K \, dArea = \sum \varphi_i - \pi.$$

Proof. The edges are geodesics, and $\varphi_i = \pi - \theta_i$, so

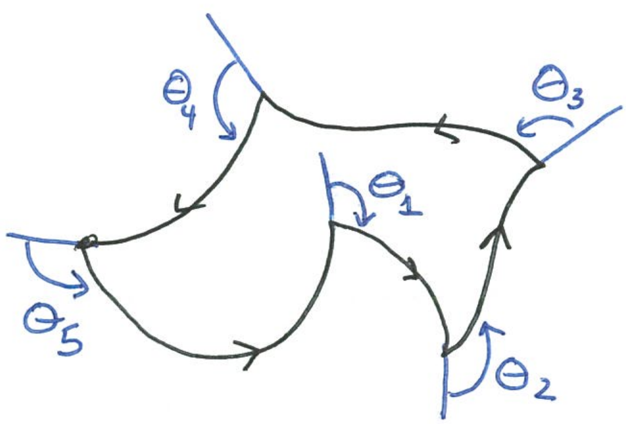
$$\iint K \, dA + \sum \theta_i = 2\pi$$

$$\Rightarrow \iint K \, dA = 2\pi - \sum \pi - \varphi_i = \sum \varphi_i - \pi. \quad \square$$

Example. Any simple, closed curve with total geodesic curvature zero on the sphere (bounds area 2π) divides the sphere into equal pieces.

Example of example. A great circle!

We can extend our theorem to curves with corners by defining turning angle as the (oriented) angle between tangents at a corner.



Theorem. If R is parametrized by a simply connected region with ∂R having corners, ~~then~~ with turning angles $\theta_1, \dots, \theta_n$,

$$\int_{\partial R} k_g \, ds + \iint_R K \, dArea + \sum \theta_i = 2\pi.$$

(skeletal notes)

Spherical cap. Now it works!

Define triangulation.

χ definition.

Gauss-Bonnet.

$$\int_{\partial M} \kappa_g \, ds + \iint_M K \, dArea + \sum \epsilon_k = 2\pi \chi$$

Write down triangle sum, interpret ~~approximate~~ sums of ~~inter~~ ~~in~~ exterior angles to get E, V terms.

Conclusions.