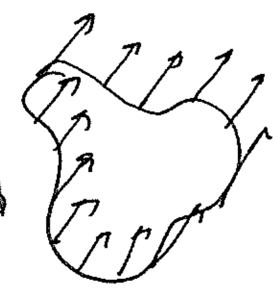
Holonomy and Gauss-Bonnet.

Definition. The holonomy of a unit vector field on a closed curve $\alpha \subset M$ is given by $\int_\alpha \langle V', \vec{n} \times V \rangle \, ds$.

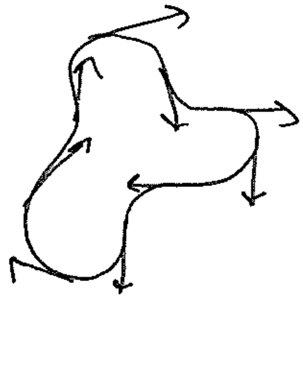
This is the ~~rate at which~~ total twisting of the frame $V, \vec{n} \times V$ for the tangent plane ~~of~~ $T_{\alpha(t)}M$ as we travel around the loop.

Examples.

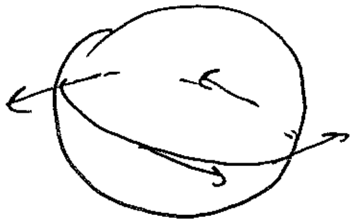
A constant vector field on a plane curve has holonomy zero.



The tangent vector on a plane curve has holonomy 2π.



The tangent vector to a great circle on the sphere has holonomy 0.



In this case V' is large, but orthogonal to $\vec{n} \times V$.

We will need to work with this in detail, so let's derive some formulas. We establish orthogonal coordinates

$X(u,v)$ so $F=0$. ~~u = constant~~

Then $\vec{e}_1 = \frac{1}{\sqrt{E}} \vec{X}_u$ and $\vec{e}_2 = \frac{1}{\sqrt{G}} \vec{X}_v$ are an orthonormal basis for T_pM. Along a curve $\alpha(t)$, let

$$\varphi_{12}(t) = \left\langle \frac{d}{dt} e_1(u(t),v(t)), e_2(u(t),v(t)) \right\rangle$$

$$= \cancel{\nabla_{\alpha(t)} \langle e_1, e_2 \rangle} \langle \nabla_{\alpha'} e_1, e_2 \rangle$$

Note that since $\langle e_1, e_2\rangle \equiv 0$,

$$\nabla_{\alpha'}\langle e_1, e_2\rangle = \langle \nabla_{\alpha'} e_1, e_2\rangle + \langle e_1, \nabla_{\alpha'} e_2\rangle = 0$$

and so $\langle e_1, \nabla_{\alpha'} e_2\rangle = -\varphi_{12}$. Further, since $\langle e_1, e_1\rangle \equiv \langle e_2, e_2\rangle \equiv 1$,

$$\langle \nabla_{\alpha'} e_1, e_1\rangle = \langle \nabla_{\alpha'} e_2, e_2\rangle = 0.$$

Proposition. In this parametrization,

$$\varphi_{12} = \frac{1}{2\sqrt{EG}}\left(-E_v u' + G_u v'\right).$$

Proof. We compute

$$\varphi_{12} = \left\langle \frac{d}{dt}\,\frac{X_u(u(t),v(t))}{\sqrt{\langle X_u, X_u\rangle}},\ \frac{X_v}{\sqrt{G}}\right\rangle$$

zero anyway!

$$= \left\langle \frac{1}{\sqrt{E}}\left(X_{uu}u' + X_{uv}v'\right) + X_u \frac{d}{dt}E^{-1/2},\ \frac{X_v}{\sqrt{G}}\right\rangle$$

$$= \frac{1}{\sqrt{EG}}\left(u'\langle X_{uu}, X_v\rangle + v'\langle X_{uv}, X_v\rangle\right)$$

Now ~~$E_v = 2\langle X_u, X_u\rangle_v = \langle X_{uv}, X$~~

Now

$$\langle X_{uu}, X_v \rangle = \langle X_u, X_v \rangle_u - \langle X_u, X_{uv} \rangle$$

$$= F_u - \frac{1}{2} E_v = -\frac{1}{2} E_v$$

0, we're in orthogonal coords!

$$\langle X_{uv}, X_v \rangle = \frac{1}{2} \langle X_v, X_v \rangle_u = \frac{1}{2} G_u, \text{ so}$$

this completes proof. □

Now we claim:

Proposition. If $\alpha(t)$: $[0, 2\pi] \to M$ is a closed curve on M, then if $\vec{X}$ is any vector in $T_{\alpha(0)} M$ and $\vec{X}(t)$ the parallel transport of $\vec{X}$, then

$$\angle (X(2\pi), X(0)) = \int_\alpha \varphi_{12}(t)\, dt.$$

Proof. Suppose $\vec{X}(t) = \cos\psi(t)\,\vec{e}_1(t) + \sin\psi(t)\,\vec{e}_2(t)$,

Then X is parallel $\Leftrightarrow$

$$0 = \nabla_{\alpha'} X = \nabla_{\alpha'} \cos\psi\, e_1 + \sin\psi\, e_2$$

$$= \cos\psi\, \nabla_{\alpha'} e_1 + \sin\psi\, \nabla_{\alpha'} e_2 + \psi'(-\sin\psi\, e_1 + \cos\psi\, e_2)$$

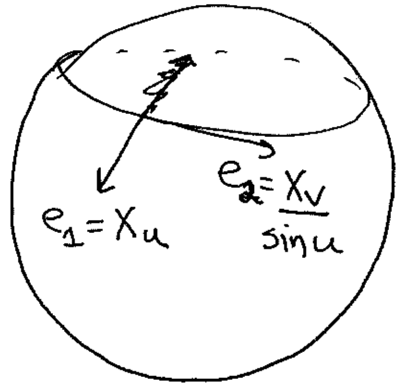
$$= \cos\psi\, \varphi_{12}\, e_2 - \sin\psi\, \varphi_{12}\, e_1 + \psi' \cos\psi\, e_2 - \psi' \sin\psi\, e_1$$

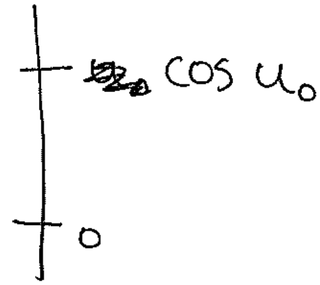
$$= \underbrace{(\varphi_{12} + \psi')}_{\text{so this is.}} \underbrace{(\cos\psi\, e_2 - \sin\psi\, e_1)}_{\text{this is not zero}}.$$

Thus $\psi'(t) = -\varphi_{12}$, and $\psi(2\pi) - \psi(0) = \int_\alpha \psi'\, dt$

$$= -\int_\alpha \varphi_{12}\, dt.$$

Example.





Recall from our parallel transport example that $\nabla_{\alpha'} e_1 = \nabla_{\alpha'} X_u = (X_{uv})^{\parallel} = \Gamma^u_{uv} X_u + \Gamma^v_{uv} X_v$

$$= \cot u_0 \, X_v = \cot u_0 \sin u_0 \, e_2$$

$$= \cos u_0 \, e_2$$

Thus $\varphi_{12} = \cos u_0$. So the holonomy around the loop is

$$-\int_0^{2\pi} \cos u_0 \, dt = -2\pi \cos u_0 .$$

Now suppose that $\alpha(s)$ is arclength parametrized, $\alpha'(s) = \cos\theta(s)\, e_1 + \sin\theta(s)\, e_2$.

Proposition. The geodesic curvature of $\alpha(s)$

$$K_g(s) = \varphi_{12}(s) + \theta'(s) = \frac{1}{2\sqrt{EG}}\left(-E_v u' + G_u v'\right) + \theta'(s).$$

Proof. We know

$$K_g = \langle \kappa N, \vec{n} \times T \rangle = \langle T', \vec{n} \times T \rangle$$

$$= \langle T', -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$$

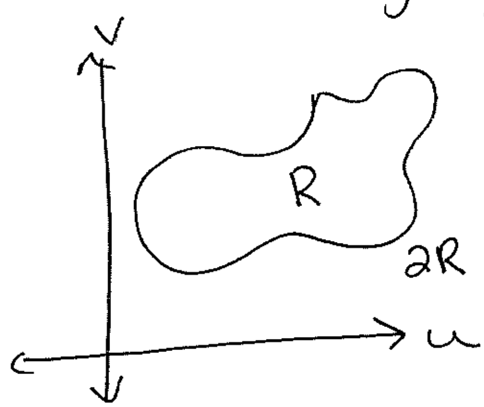
$$= \langle \nabla_{\alpha'} (\cos\theta\, e_1 + \sin\theta\, e_2), -\sin\theta\, e_1 + \cos\theta\, e_2 \rangle$$

$$= \Big\langle \cos\theta \overset{\varphi_{12} e_2}{\overset{\shortparallel}{\nabla_{\alpha'} e_1}} + \sin\theta \overset{-\varphi_{12} e_1}{\overset{\shortparallel}{\nabla_{\alpha'} e_2}} + -(\sin\theta)\theta' e_1 + (\cos\theta)\theta' e_2,\ -\sin\theta\, e_1 + \cos\theta\, e_2 \Big\rangle$$

$$= \cos^2\theta\, \varphi_{12} + \sin^2\theta\, \varphi_{12} + \sin^2\theta\, \theta' + \cos^2\theta\, \theta'$$

$$= \varphi_{12} + \theta'. \qquad \square$$

Now we're going to use a cool trick.



$$\int_{\partial R} \left\langle \begin{bmatrix} P \\ Q \end{bmatrix}, \alpha'(t) \right\rangle dt$$

$$= \iint_R Q_u - P_v \, du\, dv$$

Green's theorem (or the circulation theorem) relates $\int_{\partial R} \langle V, ds \rangle = \int_R \nabla \times V \, dArea$.

but we <u>have</u>

$$\varphi_{12}(s) = \left\langle \frac{1}{2\sqrt{EG}} \begin{bmatrix} -E_v \\ G_u \end{bmatrix}, \alpha'(t) \right\rangle$$

so this <u>must</u> be equal to

$$\iint_R \frac{1}{2}\left[\left(\frac{+E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right] du\,dv$$

$$= \iint_R \underbrace{\frac{1}{2\sqrt{EG}}\left[\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right]}_{\text{this looks familar...}} \sqrt{EG}\,du\,dv.$$

Gauss curvature! For reals!

$$= -\iint_R K \, d\text{Area}$$