A NEW COHOMOLOGICAL FORMULA FOR HELICITY IN $\mathbb{R}^{2k+1}$ REVEALS THE EFFECT OF A DIFFEOMORPHISM ON HELICITY

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ABSTRACT. The helicity of a vector field is a measure of the average linking of pairs of integral curves of the field. Computed by a six-dimensional integral, it is widely useful in the physics of fluids. For a divergence-free field tangent to the boundary of a domain in 3-space, helicity is known to be invariant under volume-preserving diffeomorphisms of the domain that are homotopic to the identity. We give a new construction of helicity for closed $(k+1)$-forms on a domain in $(2k+1)$-space that vanish when pulled back to the boundary of the domain. Our construction expresses helicity in terms of a cohomology class represented by the form when pulled back to the compactified configuration space of pairs of points in the domain. We show that our definition is equivalent to the standard one. We use our construction to give a new formula for computing helicity by a four-dimensional integral. We provide a Biot-Savart operator that computes a primitive for such forms; utilizing it, we obtain another formula for helicity. As a main result, we find a general formula for how much the value of helicity changes when the form is pushed forward by a diffeomorphism of the domain; it relies upon understanding the effect of the diffeomorphism on the homology of the domain and the de Rham cohomology class represented by the form. Our formula allows us to classify the helicity-preserving diffeomorphisms on a given domain, finding new helicity-preserving diffeomorphisms on the two-holed solid torus and proving that there are no new helicity-preserving diffeomorphisms on the standard solid torus. We conclude by defining helicities for forms on submanifolds of Euclidean space. In addition, we provide a detailed exposition of some standard 'folk' theorems about the cohomology of the boundary of domains in $\mathbb{R}^{2k+1}$.

1. Introduction

The linking number of a pair of closed curves a and b in $\mathbb{R}^3$ is a topological measure of their entanglement. We can define the linking number as the degree of the Gauss map $g\colon S^1 \times S^1 \to S^2$ given by $g(\theta,\phi) = (a(\theta) - b(\phi)) \,/\, |a(\theta) - b(\phi)|$. This degree can be written combinatorially, by counting signed crossings of a and b, but we can also write this degree as an integral by pulling back the area form on S^2 via the Gauss map and integrating over the torus $S^1 \times S^1$. This "Gauss integral formula" for linking

1991 *Mathematics Subject Classification.* Primary: 57R25; Secondary: 82D10,82D15.
Key words and phrases. helicity of vector fields, configuration spaces, Bott-Taubes integration.

number yields

$$\operatorname{Lk}(a,b) = \frac{1}{\operatorname{vol}(S^2)} \int a'(\theta) \times b'(\phi) \cdot \frac{a(\theta) - b(\phi)}{|a(\theta) - b(\phi)|} d\theta\, d\phi.$$

The linking number is a knot invariant, so it is invariant under any ambient isotopy of $\mathbb{R}^3$ carrying the curves to new curves $\tilde{a}$ and $\tilde{b}$.

Given a divergence-free vector field V on a domain $\Omega \subset \mathbb{R}^3$, we can define an analogous integral invariant known as *helicity*. The six-dimensional helicity integral, which measures the average linking number of pairs of integral curves of the field [1], is given by:

$$\mathrm{H}(V) = \frac{1}{\operatorname{vol}(S^2)} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x - y}{|x - y|^3} \ \mathrm{dvol}_x\ \mathrm{dvol}_y \tag{1}$$

Just as the linking number of a pair of curves is a knot invariant, we might expect the helicity of a vector field to be a diffeomorphism invariant. This is not always true, as we will demonstrate below, but it is true in enough cases to make helicity an important quantity in fluid dynamics and plasma physics [18].

The helicity invariant for vector fields was used in plasma physics as early as 1958 by L. Woltjer [24]. Woltjer showed that helicity was an invariant of the equations of ideal magnetohydrodynamics for an isolated system, and as such it was immediately useful in the study of astrophysical plasmas. J.J. Moreau in 1963 [19] first used the invariant to study fluid dynamics. In an influential 1969 paper [17], Keith Moffatt proved that helicity is an invariant of the equations of ideal fluid flow, even in the presence of an external force on the fluid.[1] The same invariant was associated to foliations by Godbillon and Vey in 1970 [12], by defining the foliation as the kernel of a 1-form and measuring the helicity of the form. In 1973, V.I. Arnol'd defined helicity for 2-forms in a 3-manifold [1] (the paper was published in English translation in 1986). His may be the first proof of the invariance of helicity under arbitrary volume-preserving diffeomorphisms (on a simply-connected domain).[2]

The most general invariance theorem for helicity known is:

Theorem 1.1 (Invariance of helicity theorem, [2,8])**.** *The helicity of a divergence-free vector field on $\Omega \subset \mathbb{R}^3$ is invariant under any volume-preserving diffeomorphism of Ω which is homotopic to the identity. If Ω is simply connected or the vector field is fluxless, helicity is invariant under any volume-preserving diffeomorphism of Ω.*

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[1] In the same paper, Moffatt derived a lower bound on the kinetic energy of a volume of flow with fixed helicity. Unlike Moreau, Moffatt immediately realized the importance of his results. The combination of these two theorems had a vast number of useful consequences for the dynamics of perfect fluids.

[2] A corresponding theorem for the Godbillon-Vey invariant appears in a paper of G. Raby from 1988 [20].

This theorem leaves open some natural questions: are these all of the helicity-preserving diffeomorphisms? If not, can we classify the helicity-preserving diffeomorphisms, or understand the effect of other diffeomorphisms on helicity?

To answer these questions, we notice that in the theory developed so far, there is an asymmetry between linking number and helicity – while there are several useful ways of writing linking number, including a "purely homological" expression as the degree of a map and a combinational expression as the sum of signed crossing numbers as well as an integral expression, so far the helicity has only been expressed as an integral. In this paper we try to restore the balance between linking number and helicity by providing a purely cohomological definition for the helicity of $(k+1)$-forms on domains in $\mathbb{R}^{2k+1}$ (Definition 2.12).

Here is a summary of our construction for helicity in a simple special case. Suppose that on a ball $\Omega \subset \mathbb{R}^3$ we consider the helicity of a divergence-free vector field V that is tangent to $\partial\Omega$. If we associate a 2-form α to V by pairing V with the volume form in $\mathbb{R}^3$, we will prove that the helicity can be expressed as an integral over the 6-dimensional compactified configuration space $C_2[\Omega]$ of disjoint pairs of points in Ω.

Let us understand the topology of this configuration space. We note that $C_2[\Omega]$ is homeomorphic to $\Omega \times (\Omega - B_r(x))$, where $B_r(x)$ is a small neighborhood of a point in Ω. Since Ω is a ball, $\Omega - B_r(x) \simeq S^2 \times I$ and this space is $D^3 \times S^2 \times I \simeq D^4 \times S^2$. The 2-form α can be pulled back to a pair of 2-forms α_x and α_y on $C_2[\Omega]$ under the projections of $\{x, y\}$ to x and y. We will show that $\alpha_x \wedge \alpha_y$ is a closed 4-form on $C_2[\Omega]$ which vanishes when pulled back to $\partial C_2[\Omega]$. Hence $\alpha_x \wedge \alpha_y$ will represent a class $h[g]$ in the 1-dimensional relative de Rham cohomology group $H^4(D^4 \times S^2, \partial(D^4 \times S^2))$ where g is a generator. We show that if $[g]$ is the Poincaré dual of the standard area form on S^2, then h is the helicity of V divided by the square of the volume of Ω. This gives a cohomological definition of helicity.

This definition has several attractive consequences. First, it extends helicity to higher dimensions, namely to $(k+1)$-forms on $(2k+1)$-dimensional domains. In Proposition 2.13 we show that for $(4n+1)$-dimensional domains (that is, for even values of k) helicity can extend only to a function that is identically zero. But for $(4n+3)$-dimensional domains, helicity is a nontrivial invariant for differential forms. We will see immediately that for forms, the question of whether a given diffeomorphism is volume-preserving has no bearing on whether the diffeomorphism preserves helicity. We will then give a quick proof of some of the standard invariance results for helicity in Proposition 3.1. A "combinatorial" definition of helicity as a four-dimensional integral follows in Proposition 3.3.

Following Arnold's three-dimensional definition of helicity, we get the helicity for a form on a simply-connected domain by integrating the wedge product of a $(k+1)$-form α with a k-form primitive for α. In general, not every primitive for α will produce helicity, but we give a construction for an appropriate primitive and prove that the resulting integral is helicity in Proposition 4.4.

With all this technology in hand, we then turn to our main application of these new ideas: computing the effect of an arbitrary diffeomorphism $f\colon \Omega \to \Omega'$ on the helicity of a closed $(k+1)$-form α that vanishes on the boundary of Ω.

It is a standard fact (see Appendix B, Theorems B.2 and B.3) that the k-th homology of $\partial\Omega$ splits into two subspaces generated by cycles $s_1, \dots, s_n$ which bound $(k+1)$-cycles outside Ω and Poincaré dual cycles $t_1, \dots, t_n$ which bound $(k+1)$-cycles inside Ω. With respect to a corresponding basis $\langle s'_1, \dots, s'_n, t'_1, \dots, t'_n \rangle$ for the k-th homology of $\partial\Omega'$, we can write the linear map $f_*\colon H_k(\Omega) \to H_k(\Omega')$ as a block matrix

$$f_* = \left[\begin{array}{c|c} I & 0 \\ \hline (c_{ij}) & I \end{array}\right],$$

where the c_{ij} are a symmetric matrix when k is odd and a skew-symmetric matrix when k is even. We then have

Theorem 5.8. *Let Ω^{2k+1} be a subdomain of $\mathbb{R}^{2k+1}$, and $f\colon \Omega \to \Omega'$ be an orientation-preserving diffeomorphism. Consider a closed $(k+1)$-form α that vanishes on $\partial\Omega$. The change in the helicity of α under f is*

$$\mathrm{H}(\alpha') - \mathrm{H}(\alpha) = \sum_{i,j} c_{ij} \cdot \mathrm{Flux}(\alpha, \tau_i)\, \mathrm{Flux}(\alpha, \tau_j) \tag{2}$$

where the constants c_{ij} arise from the homology isomorphism induced by f on $H_k(\partial\Omega)$ as above. The $(2m+2)$-form α' is the 'push-forward' of α under f; more precisely, $\alpha' = (f^{-1})^* \alpha$ is the pullback of α under the inverse diffeomorphism.

Note that for k even (i.e., subdomains of $\mathbb{R}^{4m+1}$) the matrix c_{ij} is skew-symmetric. So Theorem 5.8 implies that helicity does not change under any diffeomorphism of Ω. This confirms our previous calculation that helicity is always zero in these dimensions.

The simplest example of this theorem is attractive and easy to understand: a diffeomorphism of a solid torus in $\mathbb{R}^3$ isotopic to j Dehn twists changes the helicity of a 2-form on the torus by j times the square of the integral of the form over a spanning disk, as in Figure 1. In general, this allows us (for k odd) to classify the helicity-preserving diffeomorphisms from Ω to Ω as those maps for which the c_{ij} are all zero. If a diffeomorphism acts trivially on the homology of Ω, it is in this class if and only if it acts trivially on the homology of $\partial\Omega$ (Corollary 5.9).

We finish our paper with some discussion of directions for future research, including defining the (k,n,m)-helicity of $(k+1)$-forms on n-dimensional submanifolds of $\mathbb{R}^m$, an application of our results to computing "cross-helicities" of vector fields in two disjoint domains, and some thoughts on defining generalized helicities in a way inspired by the construction of the finite-type invariants for knots.

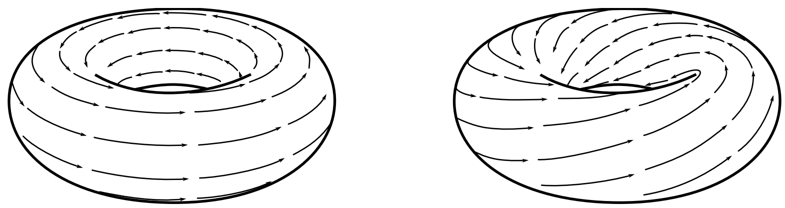

FIGURE 1. The figure shows the effect of a diffeomorphism f which applies a Dehn twist to the solid torus Ω on a vector field V dual to a 2-form α. If the radius of the tube is R, the helicity of the left hand field is 0 and the helicity of the right hand field is equal to the square of the flux of V across a spanning surface for the tube: $\pi^2 R^4$

2. Defining helicity in terms of cohomology

Helicity is motivated by the classical linking number between k and l cycles in $\mathbb{R}^{k+l+1}$. If $k = l$, observe that this linking number is only defined in odd-dimensional ambient spaces, which is why the classical linking number and helicity are defined in $\mathbb{R}^3$. In general, one could attempt to define the helicity of a tuple of k vector fields in $\mathbb{R}^{2k+1}$, applying a version of the Gauss linking integral. We find it more natural to write such a tuple as dual to a single $(k+1)$-form, which can be envisioned as the form constructed by contracting the k-tuple of vector fields with the volume form of $\mathbb{R}^{2k+1}$. In the 3-dimensional case, we take a single vector field V and construct a dual 2-form α by contracting the vector field with the volume form of $\mathbb{R}^3$ according to the rule

$$\alpha(W_1, W_2) = \mathrm{dvol}(V, W_1, W_2).$$

Under this correspondence between vector fields and forms, a divergence-free vector field tangent to the boundary of Ω becomes a closed 2-form that vanishes when pulled back to the boundary of Ω.

Start with a closed $(k+1)$-form α, defined on a domain Ω in $\mathbb{R}^{2k+1}$, that vanishes when pulled back to $\partial\Omega$. Our first goal is to define the helicity of α in terms of the cohomology class represented by a form constructed from α in the configuration space $C_2[\Omega]$ of two disjoint points in Ω. We will start by recalling the definition of $C_2[\Omega]$, which will be a smooth closed manifold with boundary and with corners, in Section 2.1. We will then give our construction of helicity in Section 2.2. Extending helicity to cases where k is odd produces a nontrivial function; when k is even, helicity extends to a function that is always zero. Finally, we will prove that our helicity is the standard helicity of a vector field in $\mathbb{R}^3$ in Section 2.3.

2.1. **The Fulton–MacPherson compactification of configuration spaces.** We start with a piece of technology: the Fulton-MacPherson compactification of a configuration space. There are many versions of this classical material (see for instance [3, 11]). We follow Sinha [21] as this gives a geometric viewpoint appropriate to our setting.

Definition 2.1. Given an m-dimensional manifold M, define the *configuration space* $C_n(M)$ to be the subspace of n-tuples $(x_i) := (x_1, \ldots, x_n) \in M^n$ such that $x_i \neq x_j$ if $i \neq j$. Let ι denote the inclusion of $C_n(M)$ in M^n.

The configuration space $C_n(M)$ may be thought of as the space of ordered n-tuples in M^n, without the diagonals. Given n distinct points in M, there are $n!$ points in $C_n(M)$ corresponding to the permutations of the n points. The Fulton–Macpherson compactification of $C_n(M)$, defined below, keeps track of the directions and relative rates of approach when configuration points come together. To simplify the definition of $C_n[M]$, we introduce a bit of notation. Let $\left[{n \atop k}\right]$ be the set of ordered k-tuples chosen from a set of n elements and $\#\left[{n \atop k}\right]$ to be the number of such tuples.

Definition 2.2 (Sinha Definition 1.3)**.** For $(i, j) \in \left[{n \atop 2}\right]$, let $\pi_{ij} \colon C_n(\mathbb{R}^m) \to S^{m-1}$ be the map which sends (x_i) to the unit vector in the direction of $x_i - x_j$. Let $[0, \infty]$ be the one-point compactification of $[0, \infty)$. For $(i, j, k) \in \left[{n \atop 3}\right]$, let $s_{ijk} \colon C_n(\mathbb{R}^m) \to [0, \infty]$ be the map which sends (x_i) to $|x_i - x_j|/|x_i - x_k|$.

We work with an arbitrary smooth manifold M by first embedding M in $\mathbb{R}^k$ (Whitney embedding), then defining the maps π_{ij} and s_{ijk} by restriction. Thus $C_n(M)$ is a submanifold of $C_n(\mathbb{R}^k)$. If $M = \mathbb{R}^k$, then it is a submanifold of itself through the identity map.

Definition 2.3 (Sinha Definition 1.3)**.** Let $A_n[M]$ be the product

$$A_n[M] = M^n \times (S^{m-1})^{\#\left[{n \atop 2}\right]} \times [0, \infty]^{\#\left[{n \atop 3}\right]}.$$

Define the *Fulton–MacPherson compactification* $C_n[M]$ to be the closure of the image of $C_n(M)$ under the map $\iota_n = \iota \times (\pi_{ij}|_{C_n(M)}) \times ((s_{ijk})|_{C_n(M)}) \colon C_n(M) \to A_n[M]$. Let $\partial C_n[M] := C_n[M] - C_n(M)$ denote the boundary of $C_n[M]$, the points that are added in the closure.

We list a few properties of this compactification, from [21] and Theorem 2.3 of [6].

Theorem 2.4. *The spaces $C_n[M]$ and $C_n(M)$ have the following properties:*

- *$C_n[M]$ is a "manifold with corners" with interior $C_n(M)$. It has the same homotopy type as $C_n(M)$. It is independent of the embedding of M in $\mathbb{R}^m$, and it is compact if M is.*
- *The inclusion of $C_n(M)$ in M^n extends to a surjective map p from $C_n[M]$ to M^n which is a homeomorphism over points in $C_n(M)$.*

- *The boundary of $C_n[M]$ is stratified into a collection of faces of various dimensions.*
- *An embedding $f\colon M \to N$ induces an embedding of manifolds with corners called the evaluation map $ev_n[f]\colon C_n[M] \to C_n[N]$ which respects the stratifications on the boundaries.*

The stratification of boundary faces of $C_n[M]$ has a beautiful combinatorial structure: in general, the set of faces of all codimensions is a Stasheff associahedron. While this structure is very interesting, we will use very little of it below, so we do not describe it in detail. We will need only the following:

Lemma 2.5. *If Ω is a k-manifold with boundary embedded in $\mathbb{R}^n$ then the boundary of the manifold $C_2[\Omega]$ has three smooth faces, $\partial\Omega \times \Omega$, $\Omega \times \partial\Omega$ and an "interior" face diffeomorphic to the unit tangent bundle $UT(\Omega)$ of Ω. Following the notation of Sinha, we will call this last face (12), meaning that points 1 and 2 come together on that face. These codimension-1 faces of the boundary of $C_2[\Omega]$ meet at faces of higher codimension.*

Proof. The space $C_2[\Omega]$ is a closed subspace of the larger space $A_2[\mathbb{R}^n]$ created by closing the image of $\Omega \times \Omega$ under the map ι. In this larger space, the boundary of the image consists of the image of $\partial(\Omega \times \Omega)$ together with a new boundary face created by removing the diagonal of $\Omega \times \Omega$.

We are only taking configurations of pairs of points in Ω, so there are no s_{ijk} maps, and only two π_{ij} maps: π_{12} and π_{21}. Along the new boundary face, then, the map ι records the location z and limiting direction u of approach of pairs of points in M. This direction, recorded by π_{12} and π_{21}, is a unit vector in the tangent space to Ω at z. The set of all such pairs is a copy of $UT(\Omega)$.

These boundary faces meet at pairs of points where, for instance, an interior point approaches a boundary point, or where both points in the pair are on the boundary of Ω. Sinha shows that these are faces of higher codimension. □

Corollary 2.6. *If Ω is a domain in $\mathbb{R}^{2k+1}$ with smooth boundary, then the boundary of $C_2[\Omega]$ consists of three faces diffeomorphic to $\Omega \times \partial\Omega$, $\partial\Omega \times \Omega$ and $\Omega \times S^{2k}$.*

Sinha additionally defines configuration spaces where one or more points in the configuration are fixed. In this case, $C_{n,k}[M]$ is the space of n points where k points are fixed and the remaining $n-k$ points vary.

2.2. **Redefining helicity.** Motivated by the Bott-Taubes approach [5] to defining finite-type knot invariants, we now seek to define helicity for a $(k+1)$-form α on a domain in R^{2k+1} by integration over an appropriate configuration space. We will construct a "universal" $2k$ form on $C_2[R^{2k+1}]$ by a Gauss map and then define helicity to be the integral of the wedge product of the universal form and a form derived from α over $C_2[\Omega]$. The corresponding approach for knots is explained beautifully by Volic [22].

So let $\Omega \subset \mathbb{R}^{2k+1}$ be a compact subdomain with piecewise smooth boundary and let α be a smooth $(k+1)$-form on Ω that is closed and vanishes on the boundary $\partial\Omega$. In $\mathbb{R}^3$, we may equivalently start with a smooth vector field V on Ω that is divergence-free and tangent to the boundary, and take α to be the dual 2-form to V. The divergence-free condition implies that α is closed, while the boundary condition on V implies that $\alpha|_{\partial\Omega} = 0$.

Lemma 2.7. *The closed $(k+1)$-form α on Ω pulls back to a pair of closed $(k+1)$-forms α_x and α_y on $C_2[\Omega]$. Hence, their wedge product $\alpha_x \wedge \alpha_y$ is a closed $(2k+2)$-form on $C_2[\Omega]$.*

Proof. We take the surjective map $p : C_2[\Omega] \to \Omega \times \Omega$, guaranteed by Theorem 2.4, and compose it with either of the two projections from $\Omega \times \Omega \to \Omega$ to obtain a map $C_2[\Omega] \to \Omega$. If we take $(x, y) \in C_2[\Omega]$, then these maps send $(x, y) \mapsto x$ or $(x, y) \mapsto y$. The pullback of α under the first map will be denoted α_x and the pullback under the second will be denoted α_y. Since α is closed, α_x, α_y and $\alpha_x \wedge \alpha_y$ are all closed forms. □

We now want to study the pullback of $\alpha_x \wedge \alpha_y$ to the boundary of $C_2[\Omega]$. To do so, we must first introduce coordinates on that boundary. As $C_2[\Omega]$ has codimension $4k$ in the ambient space $A_2[\Omega]$, the $8k+2$ natural coordinates on $A_2[\Omega]$ ($2k+1$ on Ω_x, $2k+1$ on Ω_y, $2k$ on each S^{2k}) overdetermine coordinates on $C_2[\Omega]$. In a neighborhood near the "interior" boundary face (12), which we recall is diffeomorphic to $\Omega \times S^{2k} \subset C_2[\Omega]$ by Corollary 2.6, it will be convenient to work with three different coordinate systems:

- *configuration coordinates*: $\{x_i, y_j\}$. These induce well-defined values on S^{2k} except on the face (12) where $x = y$.
- *midpoint-offset coordinates*: $\{m_i, o_j\}$. Define $m := (x + y)/2$ to be the midpoint of xy and $o := (x - y)/2$ to be the offset between x and y. These variables are defined so that $x = m + o$ and $y = m - o$. These also induce well-defined values on S^2, except on $\{o = 0\}$, which describes the boundary face (12).
- *boundary spherical coordinates*: $\{z_i, r, u_j\}$. Define $\{r, u_j\}$ as spherical coordinates on the o_j variables above so that u_j is always a unit vector and

  $$r = |o| .$$

  These have the advantage of naturally extending to the boundary face (12), described by $\{r = 0\}$.

On (12), the boundary spherical coordinates provide natural coordinates $\{z_i, u_j\}$. The $\{z_i\}$ describe the point $x = y$ while the u_j measure the limiting direction by which x and y approached each other.

Lemma 2.8. *If α vanishes when pulled back to the boundary of Ω then the form $\alpha_x \wedge \alpha_y$ vanishes when pulled back to the boundary of $C_2[\Omega]$.*

Proof. As we saw in Corollary 2.6, the boundary of $C_2[\Omega]$ consists of three codimension one faces: $\partial\Omega \times \Omega$, $\Omega \times \partial\Omega$ and (12). On the first two boundary faces, either x or y is on $\partial\Omega$. But α vanishes when pulled back to $\partial\Omega$, so α_x vanishes on $\partial\Omega_x$ and α_y on $\partial\Omega_y$. Thus $\alpha_x \wedge \alpha_y = 0$ on these faces.

The third codimension one face, which we call face (12), is all that remains. For convenience, let $I = (i_1, \ldots, i_k)$ denote a multi-index, so that $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_k}$. Using this notation, we observe that α_x can only consist of terms such as $h_I(x)dx_I$, with no y dependence. Similarly, α_y consists of terms with the same coefficient functions $h_I(y)dy_I$. The functions $h_I(x)$ are smooth functions of x since our original 2-form α on Ω was smooth.

Consider these terms on the boundary face (12), which is a copy of $\Omega \times S^{2k}$. We will write $\alpha_x \wedge \alpha_y$ in the boundary spherical coordinates $\{z_i, u_j\}$. In "midpoint-offset" coordinates, $x = m + o$ and $y = m - o$. Thus each $dx_i = dm_i + do_i$. If we now convert to boundary spherical coordinates using $o = ru$, then we see that $do_i = u_i dr + r du_i$. Now on the boundary face (12), we have $r = 0$. The coefficient functions h_I are smooth at $r = 0$, so the term $h_I r du_I = 0$ on (12) and the pullback of α_x to the boundary can have no du_i terms. Further, dr vanishes when pulled back to the boundary S^{2k}, so no dr terms can be involved either. This means that the $(k+1)$-form α_x is expressed entirely in terms of the $2k + 1$ midpoint coordinates dm_i.

But the same is true for α_y, so the $(2k + 2)$-form $\alpha_x \wedge \alpha_y$ involves only the $2k + 1$ elementary 1-forms dm_i. Thus some dm_i is repeated, forcing this form to be zero. $\square$

In the original definition of helicity in (1), we integrated over $\Omega \times \Omega$ even though the integrand was not defined on the diagonal. To justify the integration, it would be enough to show that the integrand converged on the diagonal. In fact, we can show that the integrand vanishes as we approach the diagonal. Lemma 2.8 is the appropriate version of that familiar statement in our new setting.

We now give a definition:

Definition 2.9. The *Gauss map* $g\colon C_2(\Omega) \to S^{2k}$ is given by $(x, y) \mapsto (x-y)/|x - y|$.

Lemma 2.10. *The Gauss map is a smooth map defined on all of $C_2[\Omega]$, including the boundary. The pullback of the unit volume form* vol *on S^{2k} by g defines a closed $2k$-form $g^* \operatorname{dvol}$ on $C_2[\Omega]$.*

Proof. The Gauss map extends naturally to $\Omega \times \partial\Omega$ and $\partial\Omega \times \Omega$, so we only have to worry about the boundary face (12) of $C_2[\Omega]$. But by construction, (12) is a blow-up of the diagonal of $\Omega \times \Omega$ so that the maps π_{ij} extend smoothly to the boundary. In this case, $\pi_{12} = g$, so the lemma is proven. □

This lemma demonstrates why $C_2[\Omega]$ was better for our construction than $\Omega \times \Omega$. While the latter is simpler to work with, we could not have extended the Gauss map smoothly to the diagonal of $\Omega \times \Omega$.

We can now combine the observations of Lemmas 2.7, 2.8, and 2.10 to redefine helicity. We have shown that if α is a closed form that vanishes when pulled back to the boundary of Ω, then $\alpha_x \wedge \alpha_y$ is a closed form that vanishes when pulled back to the boundary of $C_2[\Omega]$. Hence $\alpha_x \wedge \alpha_y$ represents a de Rham cohomology class $[\alpha_x \wedge \alpha_y]$ in $H^{2k+2}(C_2[\Omega], \partial C_2[\Omega]; \mathbb{R})$. Similarly, $g^* \operatorname{dvol}$ is closed so it represents a de Rham cohomology class $[g^* \operatorname{dvol}]$ in $H^{2k}(C_2[\Omega]; \mathbb{R})$. We will use de Rham cohomology (and so coefficients in $\mathbb{R}$) for the rest of the paper. We now make an observation about the volume form on $C_2[\Omega]$:

Lemma 2.11. *If M has a volume form dvol_M, then there is a natural volume form $\operatorname{dvol}_{C_2[\Omega]}$ with total volume $\operatorname{vol}(C_2[\Omega]) = \operatorname{vol}(\Omega)^2$.*

Proof. Just as we pulled back the $(k+1)$-form α to forms α_x and α_y on $C_2[\Omega]$, we can pull back $\operatorname{dvol}_\Omega$ to $(\operatorname{dvol}_\Omega)_x$ and $(\operatorname{dvol}_\Omega)_y$. Then $\operatorname{dvol}(C_2[\Omega]) = (\operatorname{dvol}_\Omega)_x \wedge (\operatorname{dvol}_\Omega)_y$. □

This lemma enables us to define helicity.

Definition 2.12. If α is a closed $(k+1)$-form on $\Omega \subset \mathbb{R}^{2k+1}$ and α vanishes when pulled back to the boundary of Ω, then we have seen that $\alpha_x \wedge \alpha_y$ defines a cohomology class in $H^{2k+2}(C_2[\Omega], \partial C_2[\Omega])$. We also know that $g^* \operatorname{dvol}_{S^{2k}}$ defines a cohomology class in $H^{2k}(C_2[\Omega])$. Let $[\operatorname{dvol}_{C_2[\Omega]}] \in H^{4k+2}(C_2[\Omega], \partial C_2[\Omega]) \simeq \mathbb{R}$ be the top class of $C_2(\Omega)$ defined by the standard volume form. The cup product $[\alpha_x \wedge \alpha_y] \cup [g^* \operatorname{dvol}_{S^{2k}}]$ is in $H^{4k+2}(C_2[\Omega])$ and is hence a multiple of $[\operatorname{dvol}_{C_2[\Omega]}]$.

We define the *helicity* $\mathrm{H}(\alpha)$ of α by

$$[\alpha_x \wedge \alpha_y] \cup [g^* \operatorname{dvol}] = \frac{\mathrm{H}(\alpha)}{\operatorname{dvol}(\Omega)^2} [\operatorname{dvol}_{C_2[\Omega]}].$$

We can calculate $\mathrm{H}(\alpha)$ explicitly as the integral

$$\mathrm{H}(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \operatorname{dvol}_{S^{2k}} . \tag{3}$$

Let $\Phi = \alpha_x \wedge \alpha_y \wedge g^* \operatorname{dvol}_{S^{2k}}$ denote the integrand above.

In Theorem 2.15, we will show that our definition agrees with the classical one on three-dimensional domains. So our definition above extends the idea of helicity to $(k+1)$-forms on $(2k+1)$-dimensional domains. For odd k values (dimensions $3, 7, 11, \dots$), helicity appears to be a nontrivial, new (for $k > 1$) diffeomorphism invariant on $(k+1)$-forms. For even k values (dimensions $5, 9, 13, \dots$), this extension produces a function which is always zero.

Proposition 2.13. *For even k values, the helicity of every $(k+1)$-form is zero.*

Proof. Let us consider the automorphism a of $C_2(\Omega)$ that interchanges x and y; it extends naturally to $C_2[\Omega]$. It reverses the orientation of $C_2[\Omega]$, since it exchanges the order of a product of odd-dimensional spaces.

We take the pullback $a^*\Phi = \alpha_y \wedge \alpha_x \wedge a^* g^* \operatorname{dvol}_{S^{2k}}$. The map a induces an antipodal map on S^{2k}; such a map has degree -1. Hence, $a^* g^* \operatorname{dvol}_{S^{2k}} = -g^* \operatorname{dvol}_{S^{2k}}$. Also, $\alpha_y \wedge \alpha_x = (-1)^{(k+1)^2} \alpha_x \wedge \alpha_y$. Combining these results, $a^*\Phi = (-1)^k \Phi$. We then compute

$$-\operatorname{H}(\alpha) = \int_{-C_2[\Omega]} \Phi = \int_{a(C_2[\Omega])} \Phi = \int_{C_2[\Omega]} a^*\Phi = \int_{C_2} (-1)^k \Phi = (-1)^k \operatorname{H}(\alpha).$$

If k is even, this implies that $\operatorname{H}(\alpha) = -\operatorname{H}(\alpha)$, i.e., that helicity is zero, and proves our proposition. If k is odd, the conclusion is a tautology: $\operatorname{H}(\alpha) = \operatorname{H}(\alpha)$. □

2.3. **Comparison with the standard definition of helicity.** This description of helicity as a cohomology class may seem quite different from the definition of helicity that we gave earlier. So before we explore the consequences of our new definition, we will reassure ourselves that this approach is correct by showing explicitly that for 2-forms defined on domains in $\mathbb{R}^3$, our 6-form Φ on $C_2(\Omega)$ is exactly the classical helicity integrand.

Lemma 2.14. *Let Ω be a compact subdomain of $\mathbb{R}^3$ with smooth boundary, and let α be a closed, smooth 2-form on Ω that vanishes on $\partial\Omega$. Let V be the vector field dual to α. Recall from Definition 2.1 that the map ι naturally embeds $C_2(\Omega)$ into $\Omega \times \Omega$. Then, the integrand Φ from (3), namely the 6-form $\alpha_x \wedge \alpha_y \wedge g^* \operatorname{dvol}$, is equal to the pullback via ι of the classical helicity integrand*

$$\frac{1}{4\pi} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} \ \operatorname{dvol}_x \operatorname{dvol}_y . \tag{4}$$

With the lemma in place, we now conclude that our definition of helicity really is the same as the standard one.

Theorem 2.15. *For three-dimensional domains, the helicity of Definition 2.12, equals the classical helicity (of equation 1). More explicitly, for a vector field V dual to a*

2-form α*,*

$$\int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol} = \frac{1}{4\pi} \int_{\Omega\times\Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} \,\mathrm{dvol}_x \,\mathrm{dvol}_y \tag{5}$$

Proof. The idea of the proof is to remove small neighborhoods of the boundary of $C_2[\Omega]$ and of the diagonal Δ in $\Omega \times \Omega$. On the removed neighborhoods, the integrals each tend to zero. On what remains, one integral is simply the pullback of the other.

Denote the integrand (4) as μ. Let U_ϵ be an ϵ-neighborhood of $\partial C_2[\Omega]$. Then,

$$\int_{C_2[\Omega]} \Phi = \int_{C_2[\Omega]-U_\epsilon} \Phi \;+\; \int_{U_\epsilon} \Phi$$

As $\epsilon \to 0$, so does $\int_{U_\epsilon} \Phi$. Then, the above lemma guarantees that $\Phi = \iota^*\mu$ on $C_2[\Omega] - U_\epsilon$. Hence,

$$\int_{C_2[\Omega]-U_\epsilon} \Phi \;=\; \int_{C_2[\Omega]-U_\epsilon} \iota^*\mu \;=\; \int_{\iota(C_2[\Omega]-U_\epsilon)} \mu$$

But, the image $\iota(C_2[\Omega] - U_\epsilon)$ is $\Omega \times \Omega$ with some neighborhood V_ϵ, dependent upon ϵ, removed. As $\epsilon \to 0$, the set V_ϵ approximates Δ.

While the integral $\int_{\Omega\times\Omega} \mu$ is improper along the diagonal, it does in fact converge. The contribution of μ integrated over neighborhoods of the diagonal converges to 0. See [8] for details.

Hence, $\int_{\iota(C_n(M)-U_\epsilon)} \iota^*\mu$ limits to the classical helicity integral $\int_{\Omega\times\Omega} \mu$. But it also limits to $\int_{C_2[\Omega]} \Phi$, so the two are equal. □

We now prove the above lemma in local coordinates at an arbitrary point in $C_2(\Omega)$.

Proof of Lemma 2.14. A choice of coordinates on Ω induces a set of configuration coordinates on $C_2(\Omega)$. At the point $p = (x, y) \in C_2(\Omega)$, we choose coordinates $\{u_i\}$ on Ω so that u_3 points along the vector $x - y$ at p. Via the map ι from Definition 2.1, these induce coordinates $\{x_i, y_i\}$ on $C_2(\Omega)$. We now calculate Φ and the classical helicity integrand 4 in these coordinates at p.

Begin by writing

$$V(x) = v_1 \frac{\partial}{\partial u_1} + v_2 \frac{\partial}{\partial u_2} + v_3 \frac{\partial}{\partial u_3} \qquad \text{and} \qquad V(y) = w_1 \frac{\partial}{\partial u_1} + w_2 \frac{\partial}{\partial u_2} + w_3 \frac{\partial}{\partial u_3}$$

so that

$$\alpha_x = v_1\, dx_2 \wedge dx_3 \;+\; v_2\, dx_3 \wedge dx_1 \;+\; v_3\, dx_1 \wedge dx_2,$$
$$\alpha_y = w_1\, dy_2 \wedge dy_3 \;+\; w_2\, dy_3 \wedge dy_1 \;+\; w_3\, dy_1 \wedge dy_2.$$

By the choice of coordinates, $\dfrac{x-y}{|x-y|^3} = \dfrac{1}{|x-y|^2}\dfrac{\partial}{\partial u_3}$. Then, the classical helicity integrand is

$$\frac{1}{4\pi}V(x)\times V(y)\cdot\frac{x-y}{|x-y|^3} = \frac{1}{4\pi}\frac{1}{|x-y|^2}(v_2w_1 - v_1w_2)\,\mathrm{dvol}_x\ \mathrm{dvol}_y\,. \tag{6}$$

Now we calculate Φ in these coordinates; we start with $g^*\,\mathrm{dvol}$, the pullback of the unit area form. Moving the configuration points in the x_3 (or y_3) direction, that is moving them closer or further apart, has no impact upon the Gauss map g, so $g^*\,\mathrm{dvol}$ cannot contain any dx_3 or dy_3 terms. Writing it in terms of the other 2-forms, we get

$$\begin{aligned} g^*\,\mathrm{dvol} =& c_1 dx_1\wedge dx_2 \ +\ c_2 dy_1\wedge dy_2 \ +\ c_3 dx_1\wedge dy_1 \ +\ c_4 dx_2\wedge dy_2 \\ &+ c_5 dx_1\wedge dy_2 \ +\ c_6 dy_1\wedge dx_2. \end{aligned}$$

But which 2-vectors on $C_n(M)$ trace out area on S^2? Neither $\frac{\partial}{\partial x_1}\wedge\frac{\partial}{\partial y_1}$ nor $\frac{\partial}{\partial x_2}\wedge\frac{\partial}{\partial y_2}$ does, so $c_3 = c_4 = 0$. The other ones do in fact trace out area on the two-sphere, but their effect must be normalized by the distance squared between x and y and by the fact that the area of the sphere integrates to 1 (we are using the unit volume form vol). Considering orientations, $c_1 = c_2 = 1/4\pi|x-y|^2 = -c_5 = -c_6$. So

$$g^*\,\mathrm{dvol} = \frac{1}{4\pi|x-y|^2}\left(dx_1\wedge dx_2 \ +\ dy_1\wedge dy_2 \ -\ dx_1\wedge dy_2 \ -\ dy_1\wedge dx_2\right).$$

We compute Φ directly:

$$\begin{aligned} \alpha_x\wedge\alpha_y\wedge g^*\,\mathrm{dvol} =& (v_1 dx_2\wedge dx_3 \ +\ v_2 dx_3\wedge dx_1) \ \wedge\ (w_1 dy_2\wedge dy_3 \ +\ w_2 dy_3\wedge dy_1) \\ &\wedge \frac{1}{4\pi}\frac{1}{|x-y|^2}\left(dx_1\wedge dx_2 \ +\ dy_1\wedge dy_2 \ -\ dx_1\wedge dy_2 \ -\ dy_1\wedge dx_2\right) \\ =& -\frac{1}{4\pi}\frac{1}{|x-y|^2}\ v_1 dx_2\wedge dx_3\wedge w_2 dy_3\wedge dy_1\wedge dx_1\wedge dy_2 \\ &- \frac{1}{4\pi}\frac{1}{|x-y|^2}\ v_2 dx_3\wedge dx_1\wedge w_1 dy_2\wedge dy_3\wedge dy_1\wedge dx_2. \end{aligned}$$

Thus, we have shown that

$$\Phi = \frac{1}{4\pi}\frac{1}{|x-y|^2}(v_2w_1 - v_1w_2)dx_1\wedge dx_2\wedge dx_3\wedge dy_1\wedge dy_2\wedge dy_3. \tag{7}$$

Pulling the classical helicity integrand (6) back via ι, we obtain Φ since $\iota^*(\mathrm{dvol}_x) = dx_1\wedge dx_2\wedge dx_3$ (and similarly for dvol_y). $\square$

3. Understanding the properties of helicity via cohomology

We have now defined helicity as a cup product of cohomology classes and have shown in the case of vector fields in $\mathbb{R}^3$ that our definition is the standard helicity integral. We now consider the consequences of our new definition and try to provide some motivation for the definition now that we have made it.

3.1. Invariance of helicity under diffeomorphisms homotopic to the identity.

In the Introduction, we discussed the development of the Helicity Invariance theorem, from the earliest versions of helicity as an invariant of ideal MHD through Arnold's picture of helicity as invariant under all diffeomorphisms on simply-connected domains to the modern picture of helicity as invariant under diffeomorphisms which are homotopic to the identity. Our redefinition of helicity allows us to give a quick proof of this invariance result.

Proposition 3.1. *Let Ω be any domain in $\mathbb{R}^{2k+1}$ and let α be a closed $(k+1)$-form on Ω that vanishes when pulled back to the boundary of Ω.*

Let $f\colon \Omega \times I \to \mathbb{R}^{2k+1}$ be a smooth map. For each fixed t, define $f_t\colon \Omega \to \Omega_t \subset \mathbb{R}^{2k+1}$ by $f_t(p) = f(p,t)$ and assume that each f_t is a diffeomorphism, with f_0 the identity map. Let $\alpha_t = (f_t^{-1})^*\alpha$ on each Ω_t.

Then $\mathrm{H}(\alpha)$ on $\Omega_0 = \Omega$ is equal to $\mathrm{H}(\alpha_1)$ on Ω_1.

Proof. There is a natural projection $\Omega \times I \to \Omega$ given by $(p,t) \mapsto p$. Pulling back under this map, we can extend α to a form on $\Omega \times I$. Similarly, there are two obvious projections $\pi_x, \pi_y\colon C_2[\Omega] \times I \to \Omega \times I$ given by $(x,y,t) \mapsto (x,t)$ and $(x,y,t) \mapsto (y,t)$. Pulling back under these maps, we can define a closed form $\alpha_x \wedge \alpha_y$ on $C_2[\Omega] \times I$.

We next define an extended Gauss map on $C_2[\Omega] \times I$ by

$$G(x,y,t) = \frac{f_t(x) - f_t(y)}{|f_t(x) - f_t(y)|}.$$

This map allows us to construct a closed $2k$-form $G^* \operatorname{dvol}_{S^{2k}}$ on $C_2[\Omega] \times I$. We note that the $(4k+2)$ helicity form $\alpha_x \wedge \alpha_y \wedge G^* \operatorname{dvol}_{S^{2k}}$ is a closed form which vanishes (by Lemma 2.8) on the boundary of $C_2[\Omega]$. By Stokes' theorem, the integral of this form over $\partial(C_2[\Omega] \times I)$ is zero. But this means that

$$\int_{C_2[\Omega]\times\{0\}} \alpha_x \wedge \alpha_y \wedge G^* \operatorname{dvol}_{S^{2k}} = \int_{C_2[\Omega]\times\{1\}} \alpha_x \wedge \alpha_y \wedge G^* \operatorname{dvol}_{S^{2k}} . \tag{8}$$

We now prove that the left hand side is $\mathrm{H}(\alpha_0)$ and the right hand side is $\mathrm{H}(\alpha_1)$.

Since f_0 is the identity map, $G(x, y, 0) = g(x, y)$ and the left hand side is clearly $\mathrm{H}(\alpha) = \mathrm{H}(\alpha_0)$. On the right-hand side, we observe that by definition

$$\mathrm{H}(\alpha_1) = \int_{C_2[\Omega_1]} (\alpha_1)_x \wedge (\alpha_1)_y \wedge g^* \,\mathrm{dvol}_{S^{2k}} = \int_{C_2[\Omega_1]} (F^{-1})^* \alpha_x \wedge (F^{-1})^* \alpha_y \wedge g^* \,\mathrm{dvol}_{S^{2k}} .$$

where $F\colon C_2[\Omega] \to C_2[\Omega_1]$ is the map of configuration spaces induced by f_1 (c.f., Theorem 2.4). We note that $G(x, y, 1) = g \circ F(x, y)$. If we pull back the integral above to $C_2[\Omega] \times \{1\}$ using F^{-1}, we get the right hand side of (8).

$$\begin{aligned} \mathrm{H}(\alpha_1) &= \int_{F^{-1}(C_2[\Omega_1]) = C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge F^* g^* \,\mathrm{dvol}_{S^{2k}} \\ &= \int_{C_2[\Omega] \times \{1\}} \alpha_x \wedge \alpha_y \wedge G^* \,\mathrm{dvol}_{S^{2k}} . \end{aligned}$$ □

3.2. **The invariance theorems for helicity and finite-type invariants.** We could have proved this theorem in a way parallel to the proof of invariance for the finite-type invariants for knots. Let $\mathfrak{Embed}(\Omega \hookrightarrow \mathbb{R}^{2k+1})$, henceforth denoted $\mathfrak{E}$, consist of all diffeomorphic embeddings of Ω into $\mathbb{R}^{2k+1}$. Maps in each connected component of $\mathfrak{E}$ are diffeotopic to one another. Define a Gauss map g_f by

$$((x, y), f) \in C_2[\Omega] \times \mathfrak{E} \mapsto \frac{f(x) - f(y)}{|f(x) - f(y)|}.$$

Consider the following diagram:

$$\begin{array}{ccc} C_2[\Omega] \times \mathfrak{Embed}(\Omega \hookrightarrow \mathbb{R}^{2k+1}) & \xrightarrow{g_f} & S^{2k} \\ \Big\downarrow \pi & & \\ \mathfrak{Embed}(\Omega \hookrightarrow \mathbb{R}^{2k+1}) & & \end{array} \tag{9}$$

where π is the natural projection in the trivial bundle $C_2[\Omega] \times \mathfrak{Embed}(\Omega \hookrightarrow \mathbb{R}^{2k+1}) \to \mathfrak{Embed}(\Omega \hookrightarrow \mathbb{R}^{2k+1})$. This is analogous to the corresponding diagram for knots introduced by Bott and Taubes [5].

Define a closed $(4k + 2)$-form

$$\Phi = \alpha_x \wedge \alpha_y \wedge g_f^*(\mathrm{dvol}) \tag{10}$$

on $C_2[\Omega] \times \mathfrak{E}$ by pulling back $\alpha_x \wedge \alpha_y$ from $C_2[\Omega]$ and the volume form dvol from S^{2k}. We now observe that integration of Φ over the fiber in the bundle $C_2[\Omega] \times \mathfrak{E} \to \mathfrak{E}$ produces a 0-form $\mathrm{H}(f)$ on $\mathfrak{E}$. The value of this 0-form on any embedding is the helicity $\mathrm{H}((f^{-1})^* \alpha)$.

Using Stokes' Theorem, we compute

$$d\,\mathrm{H}(f) = d \int_{C_2[\Omega]} \Phi = \int_{C_2[\Omega]} d\Phi - \int_{\partial C_2[\Omega]} \Phi.$$

Since Φ is a closed $(4k+2)$-form, the first summand on the right hand side vanishes. Using Lemma 2.8, we can show that Φ vanishes on $\partial C_2[\Omega]$, so the second summand on the right hand side vanishes as well. This proves that $d\,\mathrm{H}(f) = 0$, which implies that $\mathrm{H}(f)$ is constant on each connected component of $\mathfrak{E}$.

3.3. **Invariance of helicity for forms and vector fields.** The original invariance theorem for helicity of vector fields (Theorem 1.1) required that the diffeomorphisms be volume-preserving. Our theorems about the invariance of the helicity of forms, by contrast, have no such requirement.

If we fix our attention on the case $2k+1 = 3$, and consider the duality between 2-forms and vector fields, we immediately observe where the volume-preserving condition arises. Start with V dual to α on Ω and a diffeomorphism f that lies in the same component of $\mathfrak{E}$ as the identity. The helicity of α on Ω is the same as the helicity of the 2-form $\tilde{\alpha} = (f^{-1})^*(\alpha)$ on $f(\Omega)$. However if f is not volume-preserving, $\tilde{\alpha}(\cdot,\cdot)$ may not be dual to the pushforward vector field f_*V because the duality operation explicitly involves the volume form on $f(\Omega)$. Hence, differential forms produce a stronger invariance than vector fields do.

3.4. **Invariance of helicity defined with cohomologous forms.** Another interesting feature of Definition 2.12 is that the helicity of α depends only on the cohomology classes of $[\alpha_x \wedge \alpha_y]$ and $[g^* \,\mathrm{dvol}_{S^{2k}}]$. In particular, this means that we may define the helicity integrand using any volume form on S^{2k} which integrates to 1 over the sphere and get an alternate integral formula for helicity. We are motivated here by the combinatorial formula for linking number, which is derived from the Gauss integral formula for linking number by concentrating the mass of the sphere at the north pole.

Definition 3.2. Given a point $x = (x_1, x_2, x_3)$ in a domain Ω in $\mathbb{R}^3$, let $x^+(\Omega)$ be the set of points $y = (x_1, x_2, y_3) \in \Omega$ with $y_3 > x_3$.

We then have

Proposition 3.3. *The helicity of a divergence-free vector field in $\mathbb{R}^3$ which is tangent to the boundary of a domain Ω is given by the 4-dimensional integral*

$$\mathrm{H}(V) = \frac{1}{4\pi} \int_{x\in\Omega} \int_{y \in x^+(\Omega)} V(x) \cdot V(y) \times (0,0,1) \;\mathrm{dvol}_x \, dy_3.$$

Proof. Consider a sequence of 2-forms on S^2 converging to the δ-form which concentrates the area of the sphere at the north pole where each has integral 4π over the entire sphere. These forms are cohomologous as 2-forms on S^2 to the standard area form, so their pullbacks generate cohomologous 2-forms on $C_2[\Omega]$. This means that the helicities derived from the forms in the sequence are all equal to the standard helicity. But the limit of these integrals is the formula above. □

We now do an explicit helicity computation using the formula to check that it works. It is an old theorem of Moffatt [17] and Berger and Field [4] that the helicity of a divergence-free field tangent to the boundary of a pair of linked tubes is equal to the helicity of the fields in each tube plus twice the linking number of the tubes multiplied by the square of the flux of the field in the tubes (see [7] for a more general version of this theorem). Imagine then, a pair of singly-linked tubes that have rectangular cross-section with width w and height h and one overcrossing and that contain unit length fields parallel to the walls. We will assume that at the overcrossing the tubes are rectangular boxes in parallel planes, as below in Figure 2.

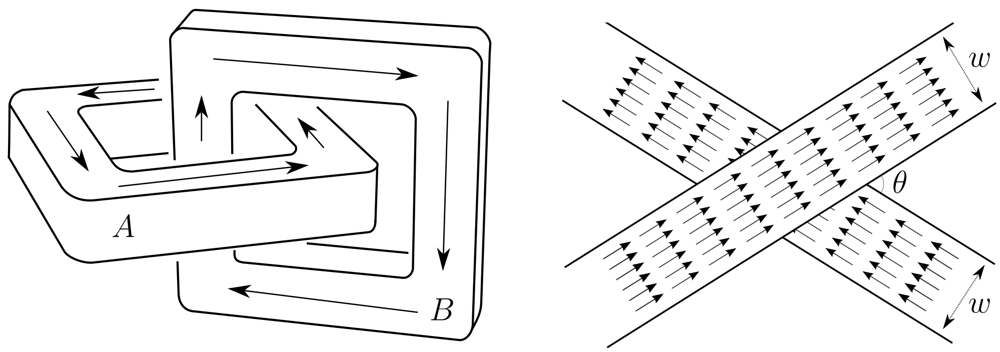

FIGURE 2. A pair of singly linked tubes A and B with rectangular cross section and a unit-length vector field tangent to the boundary of the tubes. On the right, we see the crossing from above. The crossing appears twice in the four-dimensional integral of Proposition 3.3. We are able to explicitly compute the helicity of this configuration using the proposition and show that it is equal to twice the product of the fluxes of the field in the tubes.

We can arrange the tubes so that for any pair $\{x, y\}$ with x and y in the same tube and $y \in x^+(\Omega)$, the vectors $V(x)$ and $V(y)$ are collinear. Since the integrand above vanishes for collinear vectors, these pairs will not contribute to the integral. We may further arrange the tubes so that there are only two regions where x and y are in different tubes and $y \in x^+(\Omega)$. The overcrossing pictured is one region, with x lying in the right side of ring A and y above it in the upper segment of ring B. The other region has x in the lower section of B and y in A.

We now need to integrate over these pairs. The triple product in the integrand can be rewritten $(0, 0, 1) \cdot V(x) \times V(y)$. Since $V(x)$ and $V(y)$ lie in horizontal planes in this region, the integrand always takes the constant value $\sin\theta$. On the other hand, the domain of integration (for x) is a prism of height h whose base is a parallelogram of length $w/\sin\theta$ and width w; the domain of integration for y is a line segment above each point in the x prism of length h. Thus the total (4-dimensional) volume of integration is $w^2h^2/\sin\theta$, and the value of the integral is $w^2h^2 = (wh)^2$. This is exactly the square of the flux of the vector field, and we note that the crossing is positively oriented. The other region with $y \in x^+(\Omega)$ has x in the lower section of tube B and y in tube A. This configuration is similar to the first, and makes the same contribution to the integral.

4. Helicity as a wedge product with a primitive

Arnol'd [1] defines helicity for 2-forms on simply connected 3-manifolds as the integral of the wedge product of a form α and a primitive form β with $d\beta = \alpha$. In section 2.2, we provided an alternate definition in terms of cohomology classes on configuration spaces. In this section, we reconcile these two approaches. Our efforts culminate in the next section with a formula for the change in helicity under an arbitrary diffeomorphism of Ω.

4.1. **Constructing a primitive form.** We start by observing that there is a natural fiber bundle

$$\begin{array}{ccc} C_{2,1}[\Omega] & \xrightarrow{\ i\ } & C_2[\Omega] \\ & & \downarrow{\scriptstyle \pi_x} \\ & & \Omega \end{array} \tag{11}$$

where π_x is the projection where $(x, y) \mapsto x$. Consider the $(k+1)$-form $\alpha_y = \pi_y^* \alpha$ from Lemma 2.7 generated by pulling back α from Ω in the corresponding projection π_y where $(x, y) \mapsto y$, and the $2k$-form $g^* \operatorname{dvol}$ generated by pulling back the volume form on S^{2k} under the Gauss map.

Definition 4.1. We define the *Biot-Savart operator for forms* to be the operation on $(k+1)$-forms α on Ω defined by the integration over the fiber in the bundle (11),

$$\operatorname{BS}(\alpha) = \frac{1}{\operatorname{vol}(S^{2k})} \int_{C_{2,1}[\Omega]} \alpha_y \wedge g^* \operatorname{dvol}. \tag{12}$$

We now show that the Biot-Savart operator constructs a potential for closed $(k+1)$-forms on Ω that vanish when pulled back to $\partial\Omega$.

Proposition 4.2. *If α is a closed $(k+1)$-form on Ω that vanishes when pulled back to $\partial\Omega$, then α is exact (by the Hodge decomposition theorem) and further*

$$d(\operatorname{BS}(\alpha)) = \alpha. \tag{13}$$

Proof. We will use Stokes' Theorem for fiber bundles $F^n \to E \to B$. If β is a k-form on E, then $\int_F \beta$ is a $(k-n)$-form on B, and

$$d \int_F \beta = \int_F d\beta + \int_{\partial F} \beta. \tag{14}$$

By definition, $\mathrm{BS}(\alpha)$ is the integration over the fiber $C_{2,1}[\Omega]$ of the form $\alpha_y \wedge g^* \,\mathrm{dvol}$. Since $\alpha_y \wedge g^* \,\mathrm{dvol}$ is closed on $C_{2,1}[\Omega]$, we see $d\,\mathrm{BS}(\alpha) = \int_{\partial C_{2,1}[\Omega]} \alpha_y \wedge g^* \,\mathrm{dvol}$.

Now consider the structure of the boundary of $C_{2,1}[\Omega]$. We are assuming that x is the fixed point, so the codimension-one faces consist of a copy of $\partial\Omega$ in the form of pairs (x, y) where y is on the boundary and a copy of S^{2k} where y approaches x from some direction.

On the $\partial\Omega$ face, α_y vanishes so there is no contribution to the integral. We now consider the term $\int_{S^{2k}} \alpha_y \wedge g^* \,\mathrm{dvol}$. What is this form?

In the definition of integration over the fiber (see Appendix A), we see that to integrate a $(4k + 1)$-form over a $2k$-dimensional fiber and get a resulting $(k + 1)$-form, we must write each tangent space to the total space of the bundle as a product of the $2k$-dimensional tangent space to the fiber and the tangent space to the base and decompose our $(4k + 1)$-form locally into a wedge of forms on each of these spaces. The fiber portion of the form is then integrated, while the base portion remains.

On $C_2[\Omega]$ we now establish the coordinates x_i, $z_i = y_i - x_i$, and write $z = ru$, where u is a unit vector. In the bundle (11), the base directions are the $\partial/\partial x_i$ and the fiber directions are the $\partial/\partial z_i$.

How do these coordinates extend to the boundary of the fiber? There is no difficulty in defining these coordinates on the boundary face where $y \in \partial\Omega$. But on the boundary face (12) where the two configuration points coalesce, i.e., where $r = 0$, the situation requires a bit more care. Unlike the standard polar coordinates, in which the u_i will have no meaning when $r = 0$, our compactification of the configuration space ensures that the S^{2k} defined by the u coordinates will still be present when $r = 0$.

We now consider the forms α_y and $g^* \,\mathrm{dvol}$ on the boundary face where $r = 0$ with an eye toward integration over the fiber. The form α_y is written entirely in terms of elementary forms chosen from the dy_i. But $dy_i = dx_i + dz_i$. And $dz_i = u_i dr + r du_i$, so on this face α_y is written entirely in terms of dr and the dx_i. In fact, α_y contains a precise copy of α_x together with a collection of other terms involving dr. When we pull this form back to the boundary, the dr terms vanish, leaving only a copy of α_x. On the other hand, the form $g^* \,\mathrm{dvol}$ is exactly the volume form on the boundary S^{2k}, as the Gauss map in these coordinates is just $g(x, r, u) = u$. Integrating over the fiber, we see that

$$\int_{\partial C_{2,1}[\Omega]} \alpha_y \wedge g^* \,\mathrm{dvol} = \int_{S^{2k}} \alpha_y \wedge g^* \,\mathrm{dvol} = (\mathrm{vol}\, S^{2k})\alpha_x,$$

which proves the result. $\square$

4.2. **An equivalent definition of helicity as a potential.** Motivated by Arnold's approach, can we express helicity as the integral of $\alpha \wedge \beta$ for an arbitrary primitive β of α? Unfortunately not, except in special circumstances (see [8]), since helicity is not gauge-invariant; we must choose an appropriate primitive. Below, we show that $\mathrm{BS}(\alpha)$ is an appropriate primitive and that we recover the same helicity as in Definition 2.12.

Definition 4.3 (Primitive definition of helicity)**.** Let α be a closed $(k+1)$-form which vanishes on the boundary of a compact domain Ω in $\mathbb{R}^{2k+1}$. The Hodge decomposition theorem for manifolds with boundary tells us that α is exact. Then the *"Arnol'd helicity"* of α is given by

$$\mathrm{H}(\alpha) = \int_{\Omega} \alpha \wedge \mathrm{BS}(\alpha). \tag{15}$$

The following proposition ensures that "Arnol'd helicity" is equivalent to our original definition of helicity; thus we will refer to both as *helicity*. The proof is almost immediate.

Proposition 4.4. *Given any closed $(k+1)$-form α on a domain Ω in $\mathbb{R}^{2k+1}$ which vanishes when pulled back to $\partial\Omega$, the "Arnol'd helicity" (via integrating a specific primitive)*

$$\mathrm{H}(\alpha) = \int_{\Omega} \alpha \wedge \mathrm{BS}(\alpha)$$

of Definition 4.3 is equal to the helicity (via cohomology classes)

$$\mathrm{H}(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol}$$

of Definition 2.12.

Proof. Using the bundle (11) and the properties of integration over the fiber, we see that

$$\int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol} = \int_{\Omega} \alpha_x \wedge \left(\int_{C_{2,1}[\Omega]} \alpha_y \wedge g^* \,\mathrm{dvol} \right) = \int_{\Omega} \alpha_x \wedge \mathrm{BS}(\alpha_x).$$

□

5. When is helicity invariant under a diffeomorphism?

Although it took some effort to define helicity a la Arnol'd, the payoff is substantial. We may now fully and precisely answer the question: is helicity a diffeomorphism invariant? The answer is negative, except in certain special cases (for one such case, see Proposition 3.1).

In the main result of this section, we explicitly calculate the change in helicity of α under an arbitrary diffeomorphism of Ω. Specific cases of this formula reproduce the known invariance results about helicity for domains in $\mathbb{R}^3$ (Theorem 1.1 and Proposition 3.1).

After describing the topology of domains in $\mathbb{R}^{2k+1}$, we first derive the formula for the case where Ω is a solid torus in $\mathbb{R}^3$ before describing the general result. Even though helicity is the zero function for subdomains $\Omega \subset \mathbb{R}^{4k+1}$ (i.e., k even; see Proposition 2.13), we carry out this computation in general and note the instances in which the parity of k matters. As a check, we confirm that for the k even case, helicity is invariant under all diffeomorphisms.

We begin by fixing an orientation-preserving diffeomorphism $f\colon \Omega \to \Omega'$ between domains in $\mathbb{R}^{2k+1}$. Let α be a closed $(k+1)$-form on Ω that vanishes on the boundary. Then its pullback $\alpha' = \left(f^{-1}\right)^* \alpha$ is a closed $(k+1)$-form on Ω' which vanishes on $\partial\Omega'$. By the Hodge decomposition theorem, α is exact, and hence $\left(f^{-1}\right)^* \alpha$ is also exact. Did the helicity of α change under the map f? That is, does $\mathrm{H}(\alpha)$ equal $\mathrm{H}(\alpha')$? We compute

$$\begin{aligned}
\mathrm{H}(\alpha) &= \int_\Omega \alpha \wedge \mathrm{BS}(\alpha), \\
\mathrm{H}(\alpha') &= \int_{\Omega'=f(\Omega)} \left(f^{-1}\right)^* \alpha \wedge \mathrm{BS}\left(\left(f^{-1}\right)^* \alpha\right) \\
&= \int_\Omega f^* \left(\left(f^{-1}\right)^* \alpha \wedge \mathrm{BS}(\left(f^{-1}\right)^* \alpha)\right) \\
&= \int_\Omega \alpha \wedge f^* \,\mathrm{BS}\left(\left(f^{-1}\right)^* \alpha\right).
\end{aligned}$$

Both terms integrate α wedged with a k-form, either $\mathrm{BS}(\alpha)$ or $f^* \,\mathrm{BS}(\alpha')$. Both k-forms are both primitives for α, since the exterior derivative commutes with pullbacks, i.e., $df^* \,\mathrm{BS}\left((f^{-1})^* \alpha\right) = f^* \left(d \,\mathrm{BS}\left((f^{-1})^* \alpha\right)\right) = f^* \left(\left(f^{-1}\right)^* \alpha\right) = \alpha$. However, BS does not in general commute with pullbacks, and so these two k-forms are not necessarily equal.

So we calculate the difference

$$\mathrm{H}(\alpha') - \mathrm{H}(\alpha) = \int_\Omega \alpha \wedge \left(f^* \,\mathrm{BS}(\alpha') - \mathrm{BS}(\alpha)\right) \tag{16}$$

In general terms, given two primitives β and $\tilde{\beta}$ of α, we wish to compute $\int_\Omega \alpha \wedge (\tilde{\beta} - \beta)$. We first observe that the integrand is an exact $(2k+1)$-form. In particular,

$$d\left((\tilde{\beta} - \beta) \wedge (\tilde{\beta} + \beta)\right) = 2(-1)^{k^2+2k} \alpha \wedge (\tilde{\beta} - \beta) = 2(-1)^k \alpha \wedge (\tilde{\beta} - \beta).$$

Upon simplifying this potential $2k$-form, we conclude that

$$\left((\tilde{\beta}-\beta)\wedge(\tilde{\beta}+\beta)\right)=\begin{cases}2\tilde{\beta}\wedge\beta & \text{if } k \text{ is odd,}\\ \tilde{\beta}\wedge\tilde{\beta}-\beta\wedge\beta & \text{if } k \text{ is even.}\end{cases} \tag{17}$$

Applying Stokes' theorem, we obtain

$$\begin{aligned}\int_\Omega \alpha\wedge(\tilde{\beta}-\beta) &= \int_\Omega \tfrac{1}{2}(-1)^k d\left((\tilde{\beta}-\beta)\wedge(\tilde{\beta}+\beta)\right)\\ &= (-1)^k\int_{\partial\Omega}\tfrac{1}{2}\left((\tilde{\beta}-\beta)\wedge(\tilde{\beta}+\beta)\right)\\ &= \begin{cases}\int_{\partial\Omega}\beta\wedge\tilde{\beta} & \text{if } k \text{ is odd,}\\ \frac{1}{2}\int_{\partial\Omega}\tilde{\beta}\wedge\tilde{\beta}-\beta\wedge\beta & \text{if } k \text{ is even.}\end{cases}\end{aligned}$$

Since both β and $\tilde{\beta}$ are primitives of α, and α vanishes on $\partial\Omega$, they both are closed on the boundary. On the $2k$-manifold $\partial\Omega$, the Hodge decomposition theorem tells us that every closed k-form can be written as the sum of an exact k-form and a k-form which represents a de Rham cohomology class in $H^k(\partial\Omega)$. So write

$$\beta = d\phi+\gamma, \quad \tilde{\beta}=d\tilde{\phi}+\tilde{\gamma}.$$

We now use this decomposition to analyze $\beta\wedge\tilde{\beta}$ on $\partial\Omega$. Since $\tilde{\beta}$ is closed,

$$d\phi\wedge\tilde{\beta}=d(\phi\tilde{\beta}).$$

Stokes' Theorem implies that the integral of an exact form on a boundary is zero; thus, $\int_{\partial\Omega} d\phi\wedge\tilde{\beta}=0$. Continuing this argument, we see that $\int_{\partial\Omega}\beta\wedge\tilde{\beta}=\int_{\partial\Omega}\gamma\wedge\tilde{\gamma}$. This integral is the cup product of the de Rham cohomology classes represented by γ and $\tilde{\gamma}$ in $H^k(\partial\Omega)$ evaluated on the top class of $\partial\Omega$.

Similarly, $\int_{\partial\Omega}\beta\wedge\beta=\int_{\partial\Omega}\gamma\wedge\gamma$ and $\int_{\partial\Omega}\tilde{\beta}\wedge\tilde{\beta}=\int_{\partial\Omega}\tilde{\gamma}\wedge\tilde{\gamma}$.

Viewing $\beta=\mathrm{BS}(\alpha)$ and $\tilde{\beta}=f^*\,\mathrm{BS}(\alpha')$, we may now represent the change in helicity (16) in terms of primitives that represent cohomology classes:

$$\mathrm{H}(\alpha')-\mathrm{H}(\alpha)=\begin{cases}\int_{\partial\Omega}\gamma\wedge\tilde{\gamma} & \text{if } k \text{ is odd,}\\ \frac{1}{2}\int_{\partial\Omega}\tilde{\gamma}\wedge\tilde{\gamma}-\gamma\wedge\gamma & \text{if } k \text{ is even.}\end{cases} \tag{18}$$

5.1. **Background on the homology of domains in $\mathbb{R}^{2k+1}$.** Before proceeding, we list a couple of "folk theorems" about the homology and cohomology of domains in $\mathbb{R}^{2k+1}$. To aid the non-expert reader, we also provide an example in Figure 3. We furnish proofs of these results in Appendix B. In all of these theorems, we use de Rham cohomology and so take our coefficients in $\mathbb{R}$. In this case, the Universal Coefficient Theorem gives us a natural duality isomorphism betweeen homology and cohomology. For a homology class s, we denote the dual cohomology class by s^*. We start with an existence theorem for a special basis for the k-th homology of $\partial\Omega$:

Theorem B.2. *Let Ω be a compact domain with smooth boundary in $\mathbb{R}^{2k+1}$ or S^{2k+1} (with $k > 0$) and $\bar{\Omega}$ be the complementary domain $\mathbb{R}^{2k+1} - \Omega$ or $S^{2k+1} - \Omega$. Then if we take coefficients in $\mathbb{R}$, $H_k(\partial\Omega) = H_k(\Omega) \oplus H_k(\bar{\Omega})$. Further, given any basis $\langle s_1, \dots, s_n \rangle$ for $H_k(\Omega)$ there is a corresponding basis $\langle s_1, \dots, s_n, t_1, \dots, t_n \rangle$ for $H_k(\partial\Omega)$ which we call the* Alexander basis *corresponding to $\langle s_1, \dots, s_n \rangle$ so that:*

(1) *The inclusion $\partial\Omega \hookrightarrow \Omega$ maps $\langle s_1, \dots, s_n \rangle \in H_k(\partial\Omega)$ to the original basis $\langle s_1, \dots, s_n \rangle$ for $H_k(\Omega)$ and the inclusion $\partial\Omega \hookrightarrow \bar{\Omega}$ maps $\langle t_1, \dots, t_n \rangle$ to a basis for $H_k(\bar{\Omega})$.*

(2) *$s_i = \partial\sigma_i$ for $\sigma_i \in H_{k+1}(\bar{\Omega}, \partial\bar{\Omega})$, where the σ_i form a basis for $H_{k+1}(\bar{\Omega}, \partial\bar{\Omega})$. Similarly, $t_i = \partial\tau_i$ for $\tau_i \in H_{k+1}(\Omega, \partial\Omega)$, where the τ_i form a basis for $H_{k+1}(\Omega, \partial\Omega)$.*

(3) *The cup product algebras of Ω, $\bar{\Omega}$ and $\partial\Omega$ obey*

$$s_i^* \cup \tau_j^* = \delta_{ij}[\Omega]^*, \quad t_i^* \cup \sigma_j^* = (-1)^{k+1}\delta_{ij}[\bar{\Omega}]^*$$

and

$$s_i^* \cup s_j^* = 0, \quad t_i^* \cup s_j^* = \delta_{ij}[\partial\Omega]^*, \quad t_i^* \cup t_j^* = 0.$$

(4) *The linking number $\mathrm{Lk}(s_i, t_j) = \delta_{ij}$. (Thus $\mathrm{Lk}(t_j, s_i) = (-1)^{(k+1)^2}\delta_{ij}$.)*

The Alexander duality isomorphism from $H_k(\Omega)$ to $H_k(\bar{\Omega})$ maps s_i to t_i.

We will then study the effect of a homeomorphism on the Alexander basis, proving

Theorem B.3. *Suppose that Ω and Ω' are compact domains with smooth boundary in $\mathbb{R}^{2k+1}$ or S^{2k+1} and that $f \colon \Omega \to \Omega'$ is an orientation-preserving homeomorphism. Then if $\langle s_1, \dots, s_n \rangle$ is a basis for $H_k(\Omega)$ and $\langle s'_1, \dots, s'_n \rangle$ is a corresponding basis for $H_k(\Omega')$ so that $f_*(s_i) = s'_i$, then we may build Alexander bases $\langle s_1, \dots, s_n, t_1, \dots, t_n \rangle$ for $H_k(\partial\Omega)$ and $\langle s'_1, \dots, s'_n, t'_1, \dots, t'_n \rangle$ for $H_k(\partial\Omega')$. For these bases, we have $f_*(\tau_i) = \tau'_i$ and $\partial f_*(t_i) = t'_i$ so that the map $\partial f_* \colon H_k(\partial\Omega) \to H_k(\partial\Omega')$ can be written as the $2n \times 2n$ matrix*

$$\partial f_* = \left[\begin{array}{c|c} I & 0 \\ \hline (c_{ij}) & I \end{array} \right], \tag{19}$$

where each block represents an $(n \times n)$ matrix. If k is odd, the block matrix c_{ij} is symmetric, while if k is even, the block matrix c_{ij} is skew-symmetric.

Since these theorems are somewhat complicated, we give an example in Figure 3.

5.2. **Fluxless case.**

Definition 5.1. A closed $(k+1)$-form α that vanishes on the boundary of a compact domain Ω in $\mathbb{R}^{2k+1}$ is called *fluxless* if the integral $\int_S \alpha = 0$ over every oriented $(k+1)$-cycle $S \subset \Omega$ with $\partial S \subset \partial\Omega$.

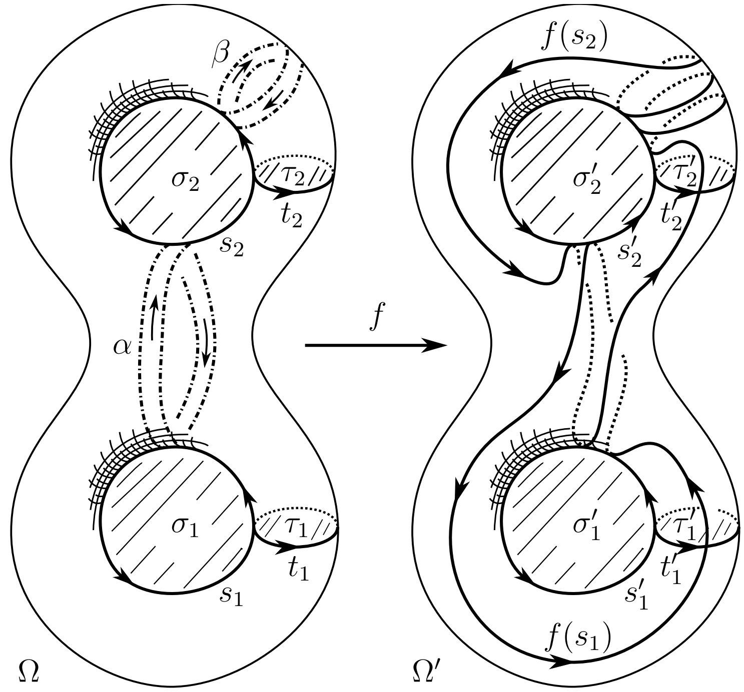

FIGURE 3. Theorem B.2 claims that there exists a special basis s_1, s_2, t_1, t_2 for $H_1(\partial\Omega)$. The left figure shows representatives for these classes. The s_i bound spanning surfaces $\sigma_i \in H_2(\bar{\Omega}, \partial\bar{\Omega})$ in the complement $\bar{\Omega}$ of Ω while the t_i bound surfaces $\tau_i \in H_2(\Omega, \partial\Omega)$. The map f is the homeomorphism from Ω to Ω' given by one Dehn twist around a disk spanning the band α and three Dehn twists around a disk spanning β. We can see the Alexander basis s_1', s_2', t_1', t_2' on Ω and the images of s_1 and s_2. (The images of t_1 and t_2 are t_1' and t_2'.) Further, if $\partial f_* \colon H_1(\partial\Omega) \to H_1(\partial\Omega')$, then we can see $\partial f_*(s_1) = s_1' - t_1' + t_2'$, $f_*(s_2) = s_2' + t_1' + 2t_2'$. This supports the claim of Theorem B.3 that if we write ∂f_* as a matrix, it has the special form of (30), and in particular that the t_2' coefficient c_{21} of $\partial f_*(s_1)$ is equal to the t_1' coefficient c_{12} of $\partial f_*(s_2)$.

We note that since the $(k+1)$-form α represents a de Rham cohomology class in the relative k-homology of Ω, the integral $\int_S \alpha$ depends only on the homology class represented by S in $H_{k+1}(\Omega, \partial\Omega)$. Since $H_{k+1}(\Omega, \partial\Omega) = H_k(\Omega)$ by Poincaré duality, if Ω has no k-homology then every $(k+1)$-form α is fluxless.

Let α be fluxless. We will utilize some facts about the cohomology and homology of the boundary of a domain in $\mathbb{R}^{2k+1}$. See Appendix B for details. First,

$$H^k(\partial\Omega) = H^k(\Omega) \oplus H^k(\mathbb{R}^{2k+1} - \Omega). \tag{20}$$

We first claim that γ and $\tilde{\gamma}$ represent classes entirely in $H^k(\Omega)$. Suppose we have a k-cycle c in $\partial\Omega$ which represents a class in $H_k(\mathbb{R}^{2k+1} - \Omega)$. Such a cycle bounds in Ω.

Since γ and β differ by $d\phi$, they have the same integral over c. Further, by Stokes' Theorem, the integral of β over c is equal to the integral of α on the $(k+1)$-cycle bounded by c in Ω. Since α is fluxless, this integral is zero. Thus $\int_c \gamma = 0$ for every k-cycle in $\partial\Omega$ which represents in $\mathbb{R}^{2k+1} - \Omega$, and (in terms of (20)), $\gamma \in H^k(\Omega)$. The same argument shows that $\tilde{\gamma} \in H^k(\Omega)$.

However, in the cup product algebra of $H^k(\partial\Omega)$, the only pairs of k-forms with non-trivial cup products have one member in $H^k(\Omega)$ and one in $H^k(\mathbb{R}^{2k+1} - \Omega)$. Hence, $\int_{\partial\Omega} \gamma \wedge \tilde{\gamma} = 0$; likewise, $\int_{\partial\Omega} \gamma \wedge \gamma = \int_{\partial\Omega} \tilde{\gamma} \wedge \tilde{\gamma} = 0$. Thus, both cases of (18) are zero, so we have proven

Proposition 5.2. *If $f\colon \Omega \to \Omega'$ is a diffeomorphism between compact domains in $\mathbb{R}^{2k+1}$ with smooth boundary, then for any fluxless $(k+1)$-form α on Ω, its helicity is invariant under f, i.e.,*

$$\mathrm{H}(\alpha) = \mathrm{H}\left(\left(f^{-1}\right)^{*} \alpha\right).$$

We note that for fluxless forms, it is not necessary to use the Biot-Savart operator in order to define helicity; replacing it with any primitive of α will produce an integral equivalent to helicity (see Definition 4.3).

But what about closed $(k+1)$-forms α which vanish on $\partial\Omega$ but are not fluxless? To understand the effect of a diffeomorphism on their helicity, we will have to compute the right hand side of (16) directly. We do so first for a solid torus before proceeding in general.

5.3. **Solid torus example.** We start with Ω, a solid torus in $\mathbb{R}^3$. Let $f\colon \Omega \to \Omega'$ be a diffeomorphism, homotopic to j Dehn twists on a spanning disk of Ω. Then f induces isomorphisms of $H_*(\Omega)$ and $H_*(\partial\Omega)$.

By (20), the boundary homology decomposes as $H_1(\partial\Omega) = H_1(\Omega) \oplus H_1(\mathbb{R}^3 - \Omega)$. We choose an *Alexander basis* (defined in Theorem B.2) $\langle s, t\rangle$: t is a meridian on $\partial\Omega$, i.e., t generates $H_1(\mathbb{R}^3 - \Omega)$; s is a longitude on $\partial\Omega$, i.e., s generates $H_1(\Omega)$. Choose s', t' similarly on $\partial\Omega'$. Let σ be a surface in $\mathbb{R}^3 - \Omega$ bounded by s; let τ be a surface in Ω bounded by t; similarly define σ' and τ'. Since f applies j Dehn twists to the solid torus Ω, we have $f_*(s) = s' + jt'$ and $f_*(t) = t'$.

We consider a closed 2-form α that vanishes on the boundary of Ω. Following the argument above, we utilize the 1-forms $\beta = \mathrm{BS}(\alpha)$ and $\tilde{\beta} = f^*\left(\mathrm{BS}(\alpha')\right)$, both primitives for α. Choose suitable 1-forms as above, γ and $\tilde{\gamma}$, which represent in terms of 1-cohomology classes. From (18), the change in helicity under f is

$$\mathrm{H}(\alpha') - \mathrm{H}(\alpha) \;\; = \;\; \int_{\partial\Omega} \gamma \wedge \tilde{\gamma}.$$

We recognize this integral as a cup product pairing in $H^1(\partial\Omega)$, since both forms in the integrand can be viewed as 1-cohomology classes. The cup product pairing is straightforward on the torus. If we write the cohomology classes dual to s and t as s^* and t^*, then by Theorem B.2

$$s^* \cup s^* = 0, \qquad t^* \cup t^* = 0, \qquad t^* \cup s^* = [\partial\Omega]^*,$$

where $[\partial\Omega]$ is the top class of the boundary in $H_2(\partial\Omega)$ and $[\partial\Omega]^*$ its dual in $H^2(\partial\Omega)$. Now we write γ and $\tilde{\gamma}$ in terms of the cohomology classes they represent,

$$[\gamma] = as^* + bt^* \qquad [\tilde{\gamma}] = \tilde{a}s^* + \tilde{b}t^*, \tag{21}$$

and find the coefficients by integrating. For example,

$$\begin{aligned}\tilde{a} = \int_s \tilde{\gamma} &= \int_s f^* \operatorname{BS}\left(\left(f^{-1}\right)^* \alpha\right) - \int_s d\tilde{\phi} = \int_{f(s)} \operatorname{BS}\left(\left(f^{-1}\right)^* \alpha\right) - 0 \\ &= \int_{s'} \operatorname{BS}(\alpha') + j \int_{t'} \operatorname{BS}(\alpha') = \int_{\sigma'} \alpha' + j \int_{\tau'} \alpha' \\ &= \int_{\sigma'} \alpha' + j \operatorname{Flux}(\alpha', \tau').\end{aligned}$$

Since α' is identically zero on $\mathbb{R}^3 - \Omega$, the first term $\int_{\sigma'} \alpha' = 0$. We also note that $\operatorname{Flux}(\alpha, \tau) = \operatorname{Flux}(\alpha', \tau')$. Thus, $\tilde{a} = j \operatorname{Flux}(\alpha)$. By similar computations, we obtain

$$[\gamma] = \operatorname{Flux}(\alpha) t^*, \qquad [\tilde{\gamma}] = \operatorname{Flux}(\alpha) t^* + j \operatorname{Flux}(\alpha) s^*.$$

Thus, we can view $\int_{\partial\Omega'} \gamma \wedge \tilde{\gamma}$ as a cup product evaluated by integration on the top class of $\partial\Omega$:

$$\operatorname{Flux}(\alpha) t^* \cup (\operatorname{Flux}(\alpha) t^* + j \operatorname{Flux}(\alpha) s^*) = j \operatorname{Flux}(\alpha)^2 [\partial\Omega]^*. \tag{22}$$

In summary, we have proven the following theorem.

Theorem 5.3. *Let Ω be a solid torus in $\mathbb{R}^3$. Let $f\colon \Omega \to \Omega'$ be an orientation-preserving map which takes Ω diffeomorphically to a subset of $\mathbb{R}^3$ and is homotopic to applying j Dehn twists to Ω. Given a closed 2-form α which vanishes on $\partial\Omega$, the change in the helicity of α under f is*

$$\operatorname{H}(\left(f^{-1}\right)^* \alpha) - \operatorname{H}(\alpha) = j \cdot \operatorname{Flux}(\alpha)^2,$$

where the flux is measured over a spanning surface in Ω which generates $H_2(\Omega, \partial\Omega)$.

This theorem lets us classify the helicity-preserving diffeomorphisms on the solid torus. We know from Proposition 3.1 that a map from the solid torus to itself preserves helicity for all 2-forms if it is homotopic to the identity through diffeomorphisms. This theorem lets us prove an (almost) converse result:

Corollary 5.4. *If $f\colon \Omega \to \Omega$ is a diffeomorphism of the solid torus to itself, then f preserves helicity for all closed 2-forms α that vanish on the boundary if and only if f is homotopic to the identity through homeomorphisms.*

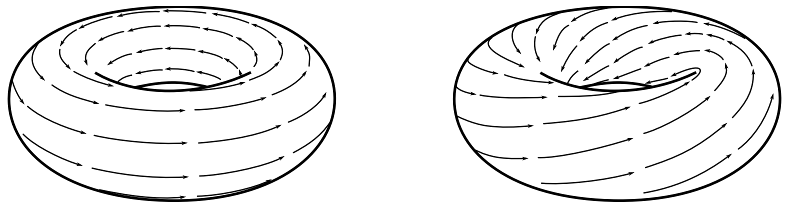

FIGURE 4. The figure shows the effect of a diffeomorphism f which applies a Dehn twist to the solid torus Ω on a vector field V dual to a 2-form α. In toroidal coordinates (r, θ, ϕ), this is the map $(r, \theta, \phi) \mapsto (r, \theta, \theta + \phi)$. The left-hand field $V = \frac{\partial}{\partial \theta}$. The field on the right is the pushforward $\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}$ of the field V under the map f. If the radius of the core circle of the torus is 1 and the radius of the tube is R, we compute that the helicity of the left hand field is 0 and the helicity of the right hand field is $\pi^2 R^4$. This agrees with Theorem 5.3.

Proof. Wainryb [23, Theorem 14] showed that the mapping class group of a solid torus is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}_2$, where the $\mathbb{Z}$ counts Dehn twists and the $\mathbb{Z}_2$ detects change of orientation. Thus a map is homotopic to the identity through homeomorphisms if and only if it preserves orientation and has no Dehn twists. By Theorem 5.3, such a map preserves helicity for any form α.

On the other hand, a map which reverses orientation reverses the sign of helicity (for any form α with nonzero helicity) and by Theorem 5.3 a map which is homotopic to a nonzero number of Dehn twists changes the helicity of any form α with nonzero flux. □

We now check this theorem with an explicit example. Suppose that Ω is the solid torus of revolution in $\mathbb{R}^3$ whose core circle has radius 1 and whose tube has radius R. We set up (standard) toroidal coordinates (r, θ, ϕ) on the torus where θ parametrizes the core circle and (r, ϕ) are polar coordinates on the cross-sections of the tube. Consider the diffeomorphism $f(r, \theta, \phi) = (r, \theta, \theta + \phi)$ on Ω. This is a volume-preserving diffeomorphism which applies one Dehn twist to Ω. We will compute the helicity of the 2-form $\alpha = *d\theta$ ($*$ is the Hodge star with respect to the standard form dvol_Ω) dual to the vector field $\frac{\partial}{\partial \theta}$ on Ω before and after the diffeomorphism f, as shown in Figure 4. It is easy to see that $f_* \frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}$. So we must compute the helicity of these two fields. It is convenient to do this via the ergodic definition of helicity given by Arnol'd [2, p.146]:

Definition 5.5. The *asymptotic linking number* of the pair of trajectories $g^t x_1$ and $g^t x_2$ $(x_1, x_2 \in \Omega)$ of a field V is defined to be the limit

$$\lambda_V(x_1, x_2) := \lim_{t_1, t_2 \to \infty} \frac{\mathrm{lk}_V(x_1, x_2; t_1, t_2)}{t_1 t_2} \tag{23}$$

where $\mathrm{lk}_V(x_1, x_2; t_1, t_2)$ is the linking number of the closures of the trajectories extending from x_1 and x_2 for times t_1 and t_2.

The definition of the asymptotic linking number requires that the trajectories be closed by a "system of short paths" joining any given pair of points on Ω and obeying certain mild technical hypotheses. Luckily, in the cases of interest to us, all of the orbits of our fields are closed with period 2π, so we can ignore these details and let

$$\lambda_V(x_1, x_2) = \lim_{p,q \in \mathbb{N} \to \infty} \frac{\mathrm{lk}_V(x_1, x_2; 2\pi p, 2\pi q)}{4\pi^2 pq}. \tag{24}$$

Now Arnol'd's ergodic definition of helicity proves that

Theorem 5.6 (Arnol'd [1,2])**.** *The average asymptotic linking number of a divergence-free field tangent to the boundary of a closed domain in $\mathbb{R}^3$ is equal to the helicity of the field. That is,*

$$\mathrm{H}(V) = \iint_{\Omega \times \Omega} \lambda_V(x_1, x_2)\ \mathrm{dvol}_{x_1}\, \mathrm{dvol}_{x_2}\,. \tag{25}$$

We can now compute the helicity of our fields. For the field $\frac{\partial}{\partial\theta}$, the orbits are all circles parallel to the xy plane. These never link, so the helicity of this field is zero.

For the field $V = \frac{\partial}{\partial\theta} + \frac{\partial}{\partial\phi}$, the orbits are all $(1, 1)$ curves on a family of nested tori foliating the solid torus Ω. Any pair of such curves has linking number 1. Now the trajectories for times $2\pi p$ and $2\pi q$ cover one of these curves p times and the other q times, so the linking number of the trajectories is pq. Taking the limit in (24), we get $\lambda_V(x_1, x_2) = 1/4\pi^2$ for all x_1, x_2 in Ω. We now compute the helicity of the field

$$\mathrm{H}(V) = \iint_{\Omega \times \Omega} \lambda_V(x_1, x_2)\, \mathrm{dvol}_{x_1}\, \mathrm{dvol}_{x_2} = \frac{\mathrm{vol}(\Omega)^2}{4\pi^2} = \frac{((2\pi)(\pi R^2))^2}{4\pi^2} = \pi^2 R^4. \tag{26}$$

Here the volume of the tube is the product of the length of the core curve and the cross-sectional area by the Tube Formula.

Now we compare the prediction of Theorem 5.3 that $\mathrm{H}(V) = \mathrm{Flux}(V)^2$. We must compute the flux of V across a cross-sectional disk of Ω. Since $\frac{\partial}{\partial\phi}$ is tangent to such a disk, the flux is the same as the flux of $\frac{\partial}{\partial\theta}$. A computation shows that this flux is πR^2, but this is not hard to see: the flux is the rate at which the disk sweeps out volume when rotated around the axis. Since this rate is constant and the disk sweeps out the entire volume $2\pi^2 R^2$ of the tube after rotation through 2π, the rate must be πR^2, as claimed. We conclude that $\mathrm{H}(V) = (\pi R^2)^2$, which agrees with our computation in (26). We note that a similar example would be easy to work out for a different number of Dehn twists, as a pair of closed orbits of the field after j Dehn twists would have linking number j.

5.4. **General formula for change of helicity.** With Theorem 5.3 in hand, we now show that a strikingly similar formula holds for general domains Ω^{2k+1}. We begin with the same setup and compute the change of helicity via (18). Throughout this section, we will make use of the Einstein summation convention.

Again, we choose an *Alexander basis* (Theorem B.2) $\langle s_1, \ldots, s_n, t_1, \ldots, t_n \rangle$ for $H_k(\partial\Omega)$. With respect to this basis, we recall that Theorem B.3 tells us that there is a corresponding Alexander basis $\langle s_1', \ldots, s_n', t_1', \ldots, t_n' \rangle$ for $H_k(\partial\Omega')$ so that the map $\partial f_* \colon H_k(\partial\Omega) \to H_k(\partial\Omega)$ looks like a $2n \times 2n$ block matrix

$$\partial f_* = \left[\begin{array}{c|c} I & 0 \\ \hline (c_{ij}) & I \end{array} \right].$$

We now write the classes $[\gamma]$ and $[\tilde{\gamma}]$ in terms of this basis.

Proposition 5.7. *In terms of the Alexander basis, the cohomology classes represented by the forms γ and $\tilde{\gamma}$ are*

$$[\gamma] = \mathrm{Flux}(\alpha, \tau_i) t_i^*, \quad [\tilde{\gamma}] = \mathrm{Flux}(\alpha, \tau_i) t_i^* + c_{ij}\, \mathrm{Flux}(\alpha, \tau_i) s_j^*$$

where the c_{ij} come from the expression of ∂f_* as a matrix above.

We obtain the general change in helicity formula.

Theorem 5.8. *Let Ω^{2k+1} be a subdomain of $\mathbb{R}^{2k+1}$, and let $f \colon \Omega \to \Omega'$ be an orientation-preserving diffeomorphism. Consider a closed $(k+1)$-form α that vanishes on $\partial\Omega$. The change in the helicity of α under f is*

$$\mathrm{H}(\alpha') - \mathrm{H}(\alpha) = \sum_{i,j} c_{ij} \cdot \mathrm{Flux}(\alpha, \tau_i)\, \mathrm{Flux}(\alpha, \tau_j) \tag{27}$$

where the constants c_{ij} arise from the homology isomorphism induced by f on $H_k(\partial\Omega)$ as above. The $(2m+2)$-form α' is the 'push-forward' of α under f; more precisely, $\alpha' = (f^{-1})^* \alpha$ is the pullback of α under the inverse diffeomorphism.

Proof of Proposition 5.7. The coefficients for $[\gamma]$ and $[\tilde{\gamma}]$ can be directly calculated. Write $[\gamma] = a_i s_i + b_i t_i$ and $[\tilde{\gamma}] = \tilde{a}_i s_i + \tilde{b}_i t_i$. Then,

$$a_i = \int_{s_i} \gamma = \int_{s_i = \partial\sigma_i} \mathrm{BS}(\alpha) - \int_{s_i} d\phi = \int_{\sigma_i} d\,\mathrm{BS}(\alpha) - 0 = \int_{\sigma_i} \alpha$$

by Stokes' Theorem. But $\alpha \equiv 0$ outside of Ω, where σ_i is located. Thus $a_i = 0$. We can similarly calculate b_i

$$b_i = \int_{\tau_i} d\,\mathrm{BS}(\alpha) - 0 = \int_{\tau_i} \alpha = \mathrm{Flux}(\alpha, \tau_i).$$

Calculating $\tilde{a}_i$ and $\tilde{b}_i$ is more involved.

$$\tilde{a}_j = \int_{s_j} \tilde{\gamma} = \int_{s_j} f^*(\mathrm{BS}((f^{-1})^* \alpha) - \int_{s_j} d\tilde{\phi} = \int_{f(s_j)} \mathrm{BS}(\alpha') - 0.$$

By (30), the image $f(s_j)$ is homologous to $s'_j + c_{ij} t'_i$. Since $\mathrm{BS}((f^{-1})^* \alpha)$ is a closed form on $\partial\Omega$, this integral is equal to

$$\begin{aligned}\tilde{a_j} &= \int_{s'_j} \mathrm{BS}(\alpha') + c_{ij} \int_{t'_i} \mathrm{BS}(\alpha') \\ &= \int_{\sigma'_j} d\,\mathrm{BS}(\alpha') + c_{ij} \int_{\tau'_i} d\,\mathrm{BS}(\alpha') \\ &= \int_{\sigma'_j} \alpha' + c_{ij} \int_{\tau'_i} \alpha'.\end{aligned}$$

Now α' vanishes outside of Ω', which is where σ'_j is located, so the first term $\int_{\sigma'_j} \alpha'$ must be zero. We compute the second integral on the original Ω,

$$\int_{\tau'_i} \alpha' = \int_{f^{-1}(\tau'_i)} \alpha$$

We know from Theorem B.3 that $f^{-1}(\tau'_i)$ is homologous to τ_i. So we conclude

$$\tilde{a_j} = c_{ij} \int_{f^{-1}(\tau'_i)} \alpha = c_{ij} \int_{\tau_i} \alpha = c_{ij} \,\mathrm{Flux}(\alpha, \tau_i).$$

We compute $\tilde{b_i}$ similarly:

$$\tilde{b_i} = \int_{t_i} \tilde{\gamma} = \int_{t_i} f^*(\mathrm{BS}((f^{-1})^* \alpha) - \int_{t_j} d\tilde{\phi} = \int_{f(t_i)} \mathrm{BS}(\alpha').$$

Since $f(t_i)$ is homologous to t'_i and $\mathrm{BS}(\alpha')$ is closed on $\partial\Omega'$,

$$\tilde{b_i} = \int_{t'_i} \mathrm{BS}(\alpha') = \int_{\tau'_i} d\,\mathrm{BS}(\alpha') = \int_{\tau'_i} \alpha' = \int_{\tau_i} \alpha = \mathrm{Flux}(\alpha, \tau_i),$$

using our previous observation that $f^{-1}(\tau'_i)$ is homologous to τ_i. This completes the proof. □

We can now derive the change in helicity formula using these coefficients. This will depend on the parity of k, so we start with the case where k is odd. The change in helicity formula (18) is

$$([\gamma] \cup [\tilde{\gamma}])([\partial\Omega]) = b_j \tilde{a_j} + a_j \tilde{b_j} = c_{ij} \,\mathrm{Flux}(\alpha, \tau_i) \,\mathrm{Flux}(\alpha, \tau_j) + 0.$$

For k even, the change in helicity formula (18) is $1/2\,([\tilde{\gamma}] \cup [\tilde{\gamma}] - [\gamma] \cup [\gamma])\,([\partial\Omega])$.

$$\begin{aligned}([\tilde{\gamma}] \cup [\tilde{\gamma}])([\partial\Omega]) &= 2\tilde{a_j}\tilde{b_j} = 2c_{ij} \,\mathrm{Flux}_i(\alpha) \,\mathrm{Flux}_j(\alpha) \\ ([\gamma] \cup [\gamma])([\partial\Omega]) &= a_i b_i = 0.\end{aligned}$$

We have calculated the change of helicity to be

$$\boxed{\mathrm{H}(\alpha') - \mathrm{H}(\alpha) = \sum_{i,j} c_{ij}\,\mathrm{Flux}(\alpha,\tau_i)\,\mathrm{Flux}(\alpha,\tau_j).} \tag{28}$$

which proves Theorem 5.8. By Theorem B.3, the matrix (c_{ij}) is skew-symmetric for k even, so the double sum on the right-hand-side of (28) vanishes in this case. Of course this is what we expect, since we know that in this case $\mathrm{H}(\alpha) = \mathrm{H}(\alpha') = 0$ by Proposition 2.13.

5.5. **Classifying the helicity-preserving diffeomorphisms.** We can now classify completely the helicity-preserving maps $f\colon \Omega \to \Omega$. For even k, Ω is $(4m+1)$-dimensional, helicity is the trivial invariant, and all maps are helicity-preserving. In the case where k is odd, Ω is $(4m+3)$-dimensional, and the helicity-preserving maps are the orientation-preserving maps with $c_{ij}\,\mathrm{Flux}(\alpha,\tau_i)\,\mathrm{Flux}(\alpha,\tau_j) = 0$ for all α. These maps form a subgroup HP of the diffeomorphism group of Ω. In Theorem B.3 the c_{ij} were determined by the homotopy type of f. Since they surely vanish for the identity map, HP also forms a subgroup of the smooth mapping class group of Ω. In this language, we can give a more standard description of the group HP.

For a surface, the Torelli subgroup of the mapping class group is the group of homeomorphisms which act trivially on homology [14]. Analogously, for any manifold M we will call the group of diffeomorphisms which act trivially on homology the Torelli subgroup Torelli(M) of the smooth mapping class group of M. Note that maps in Torelli(M) are orientation-preserving. It is easy to see

Corollary 5.9. *Since a homeomorphism from Ω to itself naturally maps $\partial\Omega$ to itself, there is a natural inclusion* $\mathrm{Torelli}(\partial\Omega) \subset \mathrm{Torelli}(\Omega)$. *With respect to this inclusion, if Ω is a domain in $\mathbb{R}^{2k+1}$ with k odd,*

$$\mathrm{HP}(\Omega) \cap \mathrm{Torelli}(\Omega) = \mathrm{Torelli}(\partial\Omega).$$

Proof. If $f \in \mathrm{Torelli}(\Omega)$, then f is orientation-preserving and Theorem 5.8 applies. The right-hand side of (27) is the action of the quadratic form defined by the matrix (c_{ij}) on the vector $\mathrm{Flux}(\alpha,\tau_i)$. Since any vector of fluxes can be obtained by choosing an appropriate α, this vanishes for all α if and only if the matrix (c_{ij}) is skew-symmetric. But since k is odd, the matrix (c_{ij}) is symmetric by Theorem B.3, so f is helicity-preserving if and only if all the c_{ij} are zero. Looking at our construction, we see that if $f \in \mathrm{Torelli}(\Omega)$, then $s_i' = s_i$. In fact, this means $t_i' = t_i$ as well. Thus f acts trivially on $H_*(\partial\Omega)$ if and only if all the c_{ij} vanish. This completes the proof. □

We note that $\mathrm{HP}(\Omega)$ is generally somewhat larger than $\mathrm{Torelli}(\partial\Omega)$: if Ω is a handlebody with n handles any (orientation-preserving) permutation of the handles will

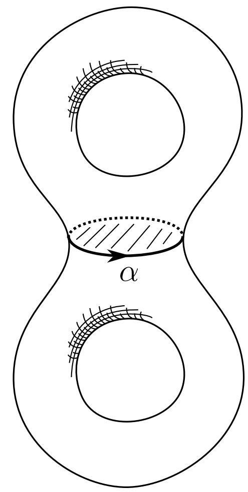

FIGURE 5. The figure shows how to construct a diffeomorphism from the solid two-holed torus Ω to itself which is helicity-preserving for all closed 2-forms on Ω which vanish on $\partial\Omega$ but not homotopic to the identity. We take a Dehn twist around the curve α on $\partial\Omega$ and extend the twist to the interior of Ω across the spanning disk shown above. The resulting map induces the identity on the homology of Ω and $\partial\Omega$, and so preserves helicity by Corollary 5.9, but it is not homotopic to the identity.

surely preserve helicity, but not act trivially on $H_*(\Omega)$ or $H_*(\partial\Omega)$. In our construction above, the matrix M will still be the identity matrix due to our careful choice of basis.

We have now given a full account of the interplay between the map f and the form α in determining the effect of a mapping on the helicity of a form. Our previous theorems are now revealed as easy corollaries of Theorem 5.8: Proposition 5.2 states that if α is fluxless, then the helicity of α is preserved by any diffeomorphism f. Indeed, the $\text{Flux}(\alpha, \tau_i)$ vanish for all i, which means that the right hand side of (28) vanishes and helicity is invariant.

On the other hand, Proposition 3.1 states that if the map f is homotopic to the identity map, then helicity is invariant under f for any form. In our new language, this is the trivial statement that the identity element in the mapping class group of Ω is in the subgroup HP. We have seen above in Corollary 5.4 that $\text{HP} = \{e\}$ for the solid torus. But in general HP is much larger. Figure 5 shows an explicit example of a helicity-preserving diffeomorphism of a domain in $\mathbb{R}^3$ which is not homotopic to the identity. Let Ω be the solid 2-holed torus, and let α be the curve around the "waist" of the torus. Our map f will be a Dehn twist around α extended to the interior of Ω along the spanning disk for α shown in the picture.

Lemma 5.10. *The map f of Figure 5 is helicity-preserving but not homotopic to the identity map.*

Proof. To see the first part, by the corollary above we must show that the map $(\partial f)_*\colon H_1(\partial\Omega) \to H_1(\partial\Omega)$ is the identity. But we can take a set of generators for $H_1(\partial\Omega)$ which are fixed by ∂f as long as we stay away from α.

To show the second, observe that if f was homotopic to the identity, then ∂f would be as well. But ∂f is a Dehn twist around an essential curve in $\partial\Omega$, so ∂f is nontrivial in the mapping class group of $\partial\Omega$ [9, Proposition 2.1]. □

6. Future Directions

Our perspective on helicity has allowed us to observe three new kinds of invariance for $\mathrm{H}(\alpha)$: invariance under change of volume form on S^{2k}, invariance in the cohomology class $[\alpha_x \wedge \alpha_y]$ in $H^{2k+2}(C_2[\Omega], \partial C_2[\Omega])$, and invariance under diffeomorphisms of Ω which preserve the homology of $\partial\Omega$. We have made stronger the analogy between helicity for forms and finite-type invariants for knots and links. And we have explained the effect of any diffeomorphism of Ω on the helicity of a form α. We devote the rest of the paper to observing some immediate consequences of our point of view, and to suggesting some future directions for further study.

6.1. **Submanifold helicities.**

So far we have only considered the case where Ω is a top-dimensional subdomain of $\mathbb{R}^{2k+1}$. We can define an analogous helicity just as easily for closed $(k+1)$-forms on an n-dimensional submanifold Ω of $\mathbb{R}^m$ that vanish on $\partial\Omega$ by

$$\mathrm{H}(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol}_{S^{m-1}} \tag{29}$$

as long as the integrand $\Phi_m = \alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol}_{S^{m-1}}$ is a $2n$-form. This requires that $2k+2+m-1 = 2n$, i.e., that $m = 2n-2k-1$. We refer to such an integral as a (k,n,m)-*helicity* and note that the helicity from Definition 2.12 is the $(k, 2k+1, 2k+1)$-helicity.

Question 6.1. *Two questions arise immediately:*

1. *For which values of k, n and m is (k,n,m)-helicity an invariant?*
2. *When is the invariant nontrivial?*

As before, we know that (k,n,m)-helicity will be an invariant if the closed form $\alpha_x \wedge \alpha_y \wedge g^* \,\mathrm{dvol}_{S^{m-1}}$ vanishes on the boundary of $C_2[\Omega]$. Following the proof of Lemma 2.7, we only have to worry about the face (12) of this boundary, which is diffeomorphic to $\Omega \times S^{m-1}$. On this face, $\alpha_x \wedge \alpha_y$ pulls back to $\alpha \wedge \alpha$. Our previous argument depended on the observation that this was a $(2k+2)$-form $\alpha \wedge \alpha$ on the $n = (2k+1)$-manifold Ω. In general, $2k+2$ may not be greater than n, so we cannot depend on this argument. However, we note that if $k+1$ is odd, then $\alpha \wedge \alpha$ vanishes by antisymmetry, providing a partial answer to the first question above. A standard example here is $(0,2,3)$-helicity, which should measure the linking of a 1-form on a surface in $\mathbb{R}^3$. We do not know whether the $(1,5,7)$-helicity measuring the linking of a 2-form on a 5-dimensional surface in $\mathbb{R}^7$ is an invariant. The $(-1,1,3)$-helicity of 0-forms on a curve in $\mathbb{R}^3$ turns out to be precisely the writhing number of the curve, so we know that this helicity is not an invariant.

What about the second question? For a contractible domain $\Omega = D^n$, the configuration space $C_2[\Omega]$ has the topology of $D^n \times D^n - \{\text{pt}\} = D^{n+1} \times S^{n-1}$ as we saw above. In this case, only the cohomology groups $H^*(C_2[\Omega], \partial C_2[\Omega])$ where $* = 0, n+1, 2n$ are nontrivial. So for a (k,n,m)-helicity to be nontrivial in this case, we must have $2k+2 = n+1$, which only occurs for $(k, 2k+1, 2k+1)$-helicity. But if Ω has nontrivial homology, then $C_2[\Omega]$ has more homology and (k,n,m)-helicity might be nontrivial. For example, we conjecture that if Ω has 1-dimensional homology, then the invariant $(0,2,3)$-helicity is nontrivial on Ω.

When k was even, we showed in Proposition 2.13 that helicity could only extend to a function that was identically zero. There is a corresponding result for (k,n,m)-helicities:

Proposition 6.2. *If $k+n$ is even, then the (k,n,m)-helicity is identically zero for any $(k+1)$-form α.*

Proof. The argument is similar to that which proves Proposition 2.13. We consider the automorphism a of $C_2(\Omega)$ that interchanges x and y; it extends naturally to $C_2[\Omega]$. It changes the orientation of $C_2[\Omega]$ by a factor of $(-1)^{n^2}$.

Next, we take the pullback $a^*\Phi_m = \alpha_y \wedge \alpha_x \wedge a^* g^* \operatorname{dvol}_{S^{m-1}}$. The map a induces an antipodal map on S^{m-1}; since m is odd, such a map has degree -1. Hence, $a^* g^* \operatorname{dvol}_{S^{m-1}} = -g^* \operatorname{dvol}_{S^{m-1}}$. Also, $\alpha_y \wedge \alpha_x = (-1)^{(k+1)^2} \alpha_x \wedge \alpha_y$. Combining these results, $a^*\Phi = (-1)^k \Phi$. We then compute

$$\int_{C_2[\Omega]} a^*\Phi = \int_{a(C_2[\Omega])} \Phi$$
$$\int_{C_2[\Omega]} (-1)^k \Phi = \int_{(-1)^n C_2[\Omega]} \Phi$$
$$(-1)^k \operatorname{H}(\alpha) = (-1)^n \operatorname{H}(\alpha)$$

If k and n have the opposite parity, this implies that $\operatorname{H}(\alpha) = -\operatorname{H}(\alpha)$, i.e., that helicity is zero, and proves our proposition. If k and n have the same parity, the conclusion is a tautology: $\operatorname{H}(\alpha) = \operatorname{H}(\alpha)$. □

6.2. **Cross-helicities.** So far we have only considered configuration spaces of two points in a single domain in $\mathbb{R}^{2k+1}$. But we could also construct similar configuration spaces where the points are restricted to lie in different domains. For instance, consider the configuration space $X \times Y$ where X and Y are disjoint linked solid tori in the form $S^k \times D^{k+1}$ in $\mathbb{R}^{2k+1}$. This configuration space simply restricts one point to lie in each torus. As before, there is a Gauss map $g\colon X \times Y \to S^{2k}$ and we can define the cross-helicity of a pair of closed $(k+1)$-forms α_x and α_y defined on X and Y and vanishing on their boundaries by

$$\mathrm{H}(\alpha_x, \alpha_y) = \int_{X\times Y} \alpha_x \wedge \alpha_y \wedge g^* \operatorname{dvol}_{S^{2k}} .$$

As before, we observe that $\alpha_x \wedge \alpha_y$ is a closed $2k+2$-form on $X \times Y$ which vanishes on the boundary of $X \times Y$. But this means that $\alpha_x \wedge \alpha_y$ represents a cohomology class in $H^{2k+2}(X \times Y)$.

Now α_x and α_y represent classes in $H^{k+1}(X, \partial X) \simeq \mathbb{R}$ and $H^{k+1}(Y, \partial Y) \simeq \mathbb{R}$. Since $X \simeq Y \simeq S^k \times D^{k+1}$, $H_{k+1}(X, \partial X)$ and $H_{k+1}(Y, \partial Y)$ are generated by cycles spanning the D^{k+1} and the classes represented by α_x and α_y are determined by their flux across these spanning cycles. Let us call the cohomology duals to the spanning cycles $[g_x]$ and $[g_y]$ so that

$$[\alpha_x] = \operatorname{Flux}(\alpha_x)[g_x] \in H^{k+1}(X, \partial X), \qquad [\alpha_y] = \operatorname{Flux}(\alpha_y)[g_y] \in H^{k+1}(Y, \partial Y).$$

We note that g_x and g_y are the Poincaré duals of the generators s_x and s_y for the homology of the S^k in X and Y with respect to the top classes in $H^{2k+1}(X)$ and $H^{2k+1}(Y)$ which integrate to 1 on their respective domains.

In $X \times Y$, the cohomology class represented by $\alpha_x \wedge \alpha_y$ is simply $\operatorname{Flux}(\alpha_x)\operatorname{Flux}(\alpha_y)[g_x] \wedge [g_y]$, which is the Poincaré dual in $X \times Y$ of $[s_x] \wedge [s_y]$. Now if we restrict the Gauss map to the core $S^k \times S^k$ of $X \times Y$, we see that the pullback of the volume form on S^{2k} represents the cohomology class $\operatorname{Lk}(X, Y)[s_x] \wedge [s_y]$. (Here Lk is the linking number of X and Y in $\mathbb{R}^{2k+1}$.)

This reproves the standard result that

$$\mathrm{H}(\alpha_x, \alpha_y) = \operatorname{Flux}(\alpha_x) \operatorname{Flux}(\alpha_y) \operatorname{Lk}(X, Y)$$

using our language for forms in linked tubes.

As in the standard helicity integral, the pullback of the area form on S^{2k} to our configuration space is a multiple of the Poincaré dual of $\alpha_x \wedge \alpha_y$. In the original helicity integral, where α_x and α_y were pulled back from the same α this multiple measured a new topological property of the form α. We could see that we were measuring new information in this case because the homology of $C_2[\Omega]$ had a new class in $H^{2k}(C_2[\Omega])$ which was not generated by the topology of the domain Ω.

On the other hand, when we calculate the cross-helicity of linked tubes, all of the homology classes involved are generated by the topology of the original domains X and Y. This means that the cross-helicity is really an invariant of the core spheres of

X and Y– the forms α_x and α_y are multiples of the Poincaré duals to the generators of these spheres, and contribute no interesting information other than their fluxes.

Similarly, several authors have defined "triple-helicity" integrals for the case of three divergence-free vector fields defined on three solid tori X, Y, and Z in space. The resulting vector field (or form) invariants turn out to be equal to

$$\operatorname{Flux}(\alpha_x)\operatorname{Flux}(\alpha_y)\operatorname{Flux}(\alpha_z) I(X,Y,Z)$$

where $I(X,Y,Z)$ is a topological invariant of the three tubes. For example, a theorem of this kind appears in Proposition 5.5 of the recent preprint of Komendarczyk [15].

We can now see that while such theorems are appealing, none of these integrals is likely to easily generalize to a meaningful invariant of 2-forms on a contractible domain in $\mathbb{R}^3$ defined by integration over $C_3[D^3]$. We could repeat the procedure above and generate a closed 6-form $\alpha_x \wedge \alpha_y \wedge \alpha_z$ which vanishes when pulled back to the boundary of $C_3[D^3]$. Unfortunately, in the 9-dimensional space $C_3[D^3]$, the cohomology group $H^6(C_3[D^3], \partial C_3[D^3]) \simeq H_3(C_3[D^3]) \simeq 0$, since the (absolute) cohomology of $C_3[D^3]$ is known to be generated by 2-forms coming from the three Gauss maps $g_{xy}(x,y,z) = {}^{x-y}/_{|x-y|}$, $g_{yz}(x,y,z) = {}^{y-z}/_{|y-z|}$ and $g_{zx}(x,y,z) = {}^{z-x}/_{|z-x|}$. Thus any such triple-helicity integral must be zero. It remains an important open problem to construct a nontrivial triple-helicity integral for forms on a contractible domain.

6.3. **Helicity and finite-type invariants.** On a last and somewhat speculative note, we wonder whether the finite-type invariants (expressed as integrals over certain configuration spaces of points on a knot) could be used to obtain integral invariants for divergence-free fields tangent to the boundary of a single knotted flux tube Ω. The values of invariants would not be interesting– we expect each to have the value $\operatorname{Flux}(\alpha) I(\Omega)$ where $I(\Omega)$ is the corresponding finite-type invariant of the tube Ω– but the integrals could in principle be used to obtain sharper energy bounds for such vector fields than the classical results of Freedman and He [10]. The major obstacle here seems to be that the construction of the finite-type invariants as integrals depends on the fact that the configuration spaces of circles are disconnected (the order of points on the circle cannot change in a connected component), allowing different components to be attached to one another to form more complicated spaces. We do not yet understand the analogous constructions for configuration spaces of points in solid tori.

7. Acknowledgements

The authors are grateful for many colleagues and friends with whom we have had fascinating and productive conversations. We are particularly indebted to Fred Cohen, Elizabeth Denne, Dennis DeTurck, Herman Gluck, Martha Holmes, Jamie Jorgensen, Will Kazez, Tom Kephart, Rafal Komendarczyk, Clint McCrory, Paul Melvin, Clay Shonkweiler, Jim Stasheff, and Shea Vick.

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Appendix A. Stokes' Theorem for Product Manifolds

Let X^n and Y^m be manifolds with boundary, where n is finite but m may be infinite. Consider the product manifold $M = X \times Y$. Let α be a smooth $(n+k)$-form on M, for $k \geq 0$. By integrating α over X, we can construct a map

$$\begin{array}{llcl} \pi_*: & \Lambda^{n+k}(X \times Y) & \rightarrow & \Lambda^k(Y), \\ \pi_*: & \alpha & \mapsto & \int_X \alpha. \end{array}$$

Here we are following Volic's notation [22] of π_*, even though this map is not a push-forward of forms; rather we map merely via integration.

Stokes' Theorem. Via this setup, the differential of the k-form $\pi_*\alpha$ on Y is

$$\begin{array}{ccccc} d\pi_*\alpha & = & \pi_* d\alpha & - & (\partial\pi)_*\alpha \\ d\int_X \alpha & = & \int_X d\alpha & - & \int_{\partial X} \alpha \end{array}$$

Rationale. Express α as the sum of three smooth forms: $\alpha = \alpha_n + \alpha_{n-1} + \beta$, where $\alpha_n = \mathrm{dvol}_x \wedge \cdots$ includes n elementary dx_i forms, α_{n-1} includes $n-1$ elementary dx_i forms, and β has less than $n-1$ elementary dx_i forms. We consider these three forms in Stokes' Theorem above. Sample terms include:

Form	Sample Term			
α_n	$f(x,y)$	dvol_x	$\wedge$	$dy_{i_1} \wedge \cdots \wedge dy_{i_k}$
α_{n-1}	$f(x,y)$	$dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge dx_n$	$\wedge$	$dy_{i_1} \wedge \cdots \wedge dy_{i_{k+1}}$
β	$f(x,y)$	$dx_1 \wedge \cdots \wedge \widehat{dx_j} \cdots \widehat{dx_k} \wedge dx_n$	$\wedge$	$dy_{i_1} \wedge \cdots \wedge dy_{i_{k+2}}$

Here the hat on $\widehat{dx_j}$ reports that term does not appear in the wedge product.

Consider the form α_n first. Note that $\int_{\partial X} \alpha_n = 0$ since α_n has greater dimension in x variables than the dimension of ∂X. Since α_n already contains the volume form on X, its differential $d\alpha_n$ will only introduce y terms, so we may differentiate under the integral sign:

$$d\int_X \alpha_n = \int_X d\alpha_n \qquad .$$

The form α_{n-1} does not contain the entire volume form on X, so $\int_X \alpha_{n-1} = 0$. When computing $\int_{\partial X} \alpha_{n-1}(x, y)$, we may hold y fixed and apply Stokes' Theorem on X to each elementary form of $(n-1)$ dx terms; thus $\int_{\partial X} \alpha_{n-1} = \int_X d\alpha_{n-1}$.

Finally, the third term β is smooth and does not yield a top-dimensional form (in terms of x variables) in any of the three integrals, so they all vanish:

$$d \int_X \beta = \int_X d\beta = \int_{\partial X} \beta = 0 \qquad .$$

For our purposes usually $k = 0$, and it is difficult to calculate $d \int_X \alpha$ directly. Thus, we shall invoke Stokes' Theorem. Usually, either α is closed or the integral of $d\alpha$ is straightforward, so our efforts concentrate upon computing $\int_{\partial X} \alpha$. Its important terms have $(n-1)$ dx terms and include a dy term. When Y is the space of embeddings or knots, we may view this term as dual to a variational vector field.

Appendix B. The homology of domains in $\mathbb{R}^{2k+1}$ (or S^{2k+1})

The purpose of this section is to give a self-contained exposition of some basic facts about the homology of domains in $\mathbb{R}^{2k+1}$ (or S^{2k+1}). We do not believe that anything in this section is original, but we do not know another reference for the details of this calculation. So we intend this section as a detailed exposition of some "folk theorems."

In what follows, we will take coefficients for homology in $\mathbb{R}$. In this case, the Universal Coefficient Theorem yields a natural duality isomorphism which pairs a homology class x with the dual cohomology class x^*. We will let $[\Omega] \in H_{2k+1}(\Omega, \partial\Omega)$ denote the top class of this orientable manifold and $[\Omega]^*$ denote the dual class in $H^{2k+1}(\Omega, \partial\Omega)$. Similarly, $[\partial\Omega] \in H_{2k}(\partial\Omega)$ will be the top class of $\partial\Omega$ and $[\partial\Omega]^*$ its dual. We will let $\tilde{H}$ denote reduced homology. We will take the linking number of two k-cycles in $\mathbb{R}^{2k+1}$ to be given by $\operatorname{Lk}(a, b) = \operatorname{Int}(a, B)$ where $b = \partial B$ and Int is the intersection number. We recall that for k-cycles, $\operatorname{Lk}(b, a) = (-1)^{(k+1)^2} \operatorname{Lk}(a, b)$ [16, Proposition 11.13].

We will also need a lemma.

Lemma B.1. *For $n < 2k$, if we think of $\mathbb{R}^{2k+1}$ as $S^{2k+1} - \{x\}$ with $x \in \operatorname{Int} \bar{\Omega}$ then for a compact domain with boundary $\Omega \subset \mathbb{R}^{2k+1}$,*

$$H_n(\mathbb{R}^{2k+1} - \Omega) = H_n(S^{2k+1} - \Omega).$$

Proof. Taking an open ball D^{2k+1} around the point x, we have written S^{2k+1} as a union of open sets. Then the (reduced) Mayer-Vietoris sequence yields an exact sequence

$$\to H_n(S^{2k}) \to H_n(D^{2k+1}) \oplus H_n(\mathbb{R}^{2k+1} - \Omega) \to H_n(S^{2k+1} - \Omega) \to H_{n-1}(S^{2k}) \to$$

Since D^{2k+1} is contractible (and we are in reduced homology), this provides the desired isomorphism immediately since the first and last homology groups in the sequence vanish. □

We start with an existence theorem for a special basis for the k-th homology of $\partial\Omega$:

Theorem B.2. *Let Ω be a compact domain with smooth boundary in $\mathbb{R}^{2k+1}$ or S^{2k+1} (with $k > 0$) and $\bar{\Omega}$ be the complementary domain $\mathbb{R}^{2k+1} - \Omega$ or $S^{2k+1} - \Omega$. Then if we take coefficients in $\mathbb{R}$, $H_k(\partial\Omega) = H_k(\Omega) \oplus H_k(\bar{\Omega})$. Further, given any basis $\langle s_1, \dots, s_n\rangle$ for $H_k(\Omega)$ there is a corresponding basis $\langle s_1, \dots, s_n, t_1, \dots, t_n\rangle$ for $H_k(\partial\Omega)$ which we call the* Alexander basis *corresponding to $\langle s_1, \dots, s_n\rangle$ so that:*

(1) *The inclusion $\partial\Omega \hookrightarrow \Omega$ maps $\langle s_1, \dots, s_n\rangle \in H_k(\partial\Omega)$ to the original basis $\langle s_1, \dots, s_n\rangle$ for $H_k(\Omega)$ and the inclusion $\partial\Omega \hookrightarrow \bar{\Omega}$ maps $\langle t_1, \dots, t_n\rangle$ to a basis for $H_k(\bar{\Omega})$.*
(2) *$s_i = \partial\sigma_i$ for $\sigma_i \in H_{k+1}(\bar{\Omega}, \partial\bar{\Omega})$, where the σ_i form a basis for $H_{k+1}(\bar{\Omega}, \partial\bar{\Omega})$. Similarly, $t_i = \partial\tau_i$ for $\tau_i \in H_{k+1}(\Omega, \partial\Omega)$, where the τ_i form a basis for $H_{k+1}(\Omega, \partial\Omega)$.*
(3) *The cup product algebras of Ω, $\bar{\Omega}$ and $\partial\Omega$ obey*
$$s_i^* \cup \tau_j^* = \delta_{ij}[\Omega]^*, \quad t_i^* \cup \sigma_j^* = (-1)^{k+1}\delta_{ij}[\bar{\Omega}]^*$$
and
$$s_i^* \cup s_j^* = 0, \quad t_i^* \cup s_j^* = \delta_{ij}[\partial\Omega]^*, \quad t_i^* \cup t_j^* = 0.$$
(4) *The linking number $\mathrm{Lk}(s_i, t_j) = \delta_{ij}$. (Thus $\mathrm{Lk}(t_j, s_i) = (-1)^{(k+1)^2}\delta_{ij}$.)*

The Alexander duality isomorphism from $H_k(\Omega)$ to $H_k(\bar{\Omega})$ maps s_i to t_i.

We will then study the effect of a homeomorphism on the Alexander basis, proving

Theorem B.3. *Suppose that Ω and Ω' are compact domains with smooth boundary in $\mathbb{R}^{2k+1}$ or S^{2k+1} and that $f\colon \Omega \to \Omega'$ is an orientation-preserving homeomorphism. Then if $\langle s_1, \dots, s_n\rangle$ is a basis for $H_k(\Omega)$ and $\langle s'_1, \dots, s'_n\rangle$ is a corresponding basis for $H_k(\Omega')$ so that $f_*(s_i) = s'_i$, then we may build Alexander bases $\langle s_1, \dots, s_n, t_1, \dots, t_n\rangle$ for $H_k(\partial\Omega)$ and $\langle s'_1, \dots, s'_n, t'_1, \dots, t'_n\rangle$ for $H_k(\partial\Omega')$. For these bases, we have $f_*(\tau_i) = \tau'_i$ and $\partial f_*(t_i) = t'_i$ so that the map $\partial f_*\colon H_k(\partial\Omega) \to H_k(\partial\Omega')$ can be written as the $2n \times 2n$ matrix*

$$M = \left[\begin{array}{c|c} I & 0 \\ \hline (c_{ij}) & I \end{array}\right], \tag{30}$$

where each block represents an $(n \times n)$ matrix. If k is odd, the block matrix c_{ij} is symmetric, while if k is even, the block matrix c_{ij} is skew-symmetric.

An example of these theorems was shown in Figure 3. We now restate some of our main tools for this theorem. The first is a form of Poincaré duality [13, Theorem 3.4.3, p.254]:

Theorem B.4 (Lefschetz Duality)**.** *Suppose M is a compact orientable n-manifold with boundary ∂M. Then cap product with a top class $[M] \in H^n(M, \partial M)$ gives isomorphisms $D_M\colon H^i(M) \to H_{n-i}(M, \partial M)$ for all k.*

We now begin the proof. We are essentially reproving the Alexander duality theorem while recording additional information along the way.

Proof of Theorem B.2. Let us restrict our attention to the case where $\Omega \subset S^{2k+1}$ for convenience. (Lemma B.1 shows that the same proof works in both cases.) We observe that since $\partial\Omega$ is compact and smooth, it has a open tubular neighborhood which deformation retracts to $\partial\Omega$. Using this, the reduced Mayer-Vietoris sequence [13, p.150] yields a long exact sequence including

$$H_{k+1}(S^{2k+1}) = 0 \to H_k(\partial\Omega) \xrightarrow{\Phi} H_k(\Omega) \oplus H_k(\bar{\Omega}) \xrightarrow{\Psi} H_k(S^{2k+1}) = 0.$$

This proves that the map Φ given by the inclusions of $\partial\Omega$ into Ω and $\bar{\Omega}$ is an isomorphism between $H_k(\partial\Omega)$ and $H_k(\Omega) \oplus H_k(\bar{\Omega})$, proving the first statement in our theorem.

Using this isomorphism, we see that our original basis $s_1, \ldots, s_n \in H_k(\Omega)$ is the Φ-image of a linearly independent set of classes $s_1, \ldots, s_n \in H_k(\partial\Omega)$ which vanish when included in $H_k(\bar{\Omega})$. By Lefschetz Duality (B.4),

$$[\Omega]\cap\colon\ H^k(\Omega) \to H_{k+1}(\Omega, \partial\Omega),$$

is an isomorphism. Let $\tau_1, \ldots, \tau_n$ denote the images of the cohomology duals of $s_1, \ldots, s_n$ under this isomorphism. By construction, $[\Omega] \cap s_i^* = \tau_i$, or $\tau_j^*([\Omega] \cap s_i^*) = \delta_{ij}$. Now for $\alpha \in H_{k+l}(X, \partial X)$, $\phi \in H^k(X)$ and $\psi \in H^l(X, \partial X)$, the cup and cap product are related by the formula [13, p.249],

$$\psi(\alpha \cap \phi) = (\phi \cup \psi)(\alpha). \tag{31}$$

Taking $\psi = \tau_j^*$, $\alpha = [\Omega]$, and $\phi = s_i^*$, we get

$$1 = \tau_j^*([\Omega] \cap s_i^*) = (s_i^* \cup \tau_j^*)([\Omega]).$$

This proves our statement about the cup structure of Ω.

We now map the τ_i to classes in $H_k(\bar{\Omega})$. If we take a small open neighborhood $\partial\Omega_o$ of $\partial\Omega$ which deformation retracts to $\partial\Omega$, we can construct $\Omega_o = \Omega \cup \partial\Omega_o$ so that the closure of $\bar{\Omega}_o = S^{2k+1} - \Omega_o$ is contained in the interior of $\bar{\Omega}$. Clearly $\Omega_o = S^{2k+1} - \bar{\Omega}_o$ and

$$\bar{\Omega} - \bar{\Omega}_o = (S^{2k+1} - \Omega) - (S^{2k+1} - (\Omega \cup \partial\Omega_o)) = \partial\Omega_o.$$

This means that the pair $(\Omega_o, \partial\Omega_o) = (S^{2k+1} - \bar{\Omega}_o, \bar{\Omega} - \bar{\Omega}_o)$. Further, since the closure of $\bar{\Omega}_o$ is contained in the interior of $\bar{\Omega}$, the inclusion of $(S^{2k+1} - \bar{\Omega}_o, \bar{\Omega} - \bar{\Omega}_o) \hookrightarrow (S^{2k+1}, \bar{\Omega})$ induces the homology isomorphism

$$H_{k+1}(\Omega, \partial\Omega) = H_{k+1}(\Omega_o, \partial\Omega_o) = H_{k+1}(S^{2k+1}, \bar{\Omega}).$$

by deformation retraction of $(\Omega_o, \partial\Omega_o)$ onto $(\Omega, \partial\Omega)$ and excision [13, Theorem 2.20, p.119]. But the exact sequence of the pair $(S^{2k+1}, \bar{\Omega})$ contains

$$H_{k+1}(S^{2k+1}) = 0 \to H_{k+1}(S^{2k+1}, \bar{\Omega}) \xrightarrow{\partial} H_k(\bar{\Omega}) \to H_k(S^{2k+1}) = 0$$

So the boundary map carries the τ_i to a set of generators t_i for $H_k(\bar{\Omega})$. The entire map we have built from s_i (as a basis for $H_k(\Omega)$) to t_i (as a basis for $H_k(\bar{\Omega})$) is the Alexander duality isomorphism.

Since we can pull these t_i back to $H_k(\partial\Omega)$ under the isomorphism $\Phi\colon H_k(\partial\Omega) \to H_k(\Omega) \oplus H_k(\bar{\Omega})$, we can regard the t_i as a linearly independent set of elements in $H_k(\partial\Omega)$ which complete the Alexander basis $\langle s_1, \ldots, s_n, t_1, \ldots, t_n\rangle$ for $H_k(\partial\Omega)$. In fact, we can choose representatives for the t_i so that $t_i = \partial\tau_i$. We now observe that $\operatorname{Lk}(s_i, t_j) = \operatorname{Int}(s_i, \tau_j) = (s_i^* \cup \tau_j^*)[\Omega] = \delta_{ij}$.

We now work out the cup product of s_i^* and t_j^*. The relation between the cap product and the boundary operator for an i-chain α in $C_i(X, A)$ and a cochain $\beta \in C^l(X)$ is given by

$$\partial(\alpha \cap \beta) = (-1)^l(\partial\alpha \cap \beta - \alpha \cap \delta\beta)$$

where δ is the coboundary operator [13, p.240]. We now compute

$$t_i = \partial\tau_i = \partial([\Omega] \cap s_i^*) = (-1)^k(\partial[\Omega] \cap s_i^* - [\Omega] \cap \delta s_i^*) = (-1)^k([\partial\Omega] \cap s_i^*), \tag{32}$$

where $\delta s_i^* = 0$ because s_i^* is a cocycle. We can compute

$$\delta_{ij} = t_j^*(t_i) = (-1)^k t_j^*([\partial\Omega] \cap s_i^*) = (-1)^k(s_i^* \cup t_j^*)[\partial\Omega]. \tag{33}$$

Now we recall that the cup product of $\alpha \in H^i(X)$ and $\beta \in H^j(X)$ obeys the (anti)commutativity relation [13, Theorem 3.14, p.215]:

$$\alpha \cup \beta = (-1)^{ij}\beta \cup \alpha.$$

Using this equation, we see immediately that (33) yields $(t_j^* \cup s_i^*)[\partial\Omega] = \delta_{ij}$ whether k is even or odd. Thus we have

$$\delta_{ij} = (t_j^* \cup s_i^*)[\partial\Omega] = s_i^*([\partial\Omega] \cap t_j^*),$$

and $[\partial\Omega] \cap t_i^* = s_i + d_l t_l$. (We will shortly argue that the d_l are all zero.)

As above, we know $[\bar{\Omega}]\cap\colon H^k(\bar{\Omega}) \to H_{k+1}(\bar{\Omega}, \partial\bar{\Omega})$ is an isomorphism. We let $\sigma_i = (-1)^{k+1}[\bar{\Omega}] \cap t_i^*$. We now show $\partial\sigma_i \in H_k(\partial\Omega) = s_i$. We notice first that $\partial\sigma_i$ is a linear combination of s_j since it is certainly the case that $\partial\sigma_i$ bounds in $\bar{\Omega}$, so $\partial\sigma_i$ must be contained in the $H_k(\Omega)$ summand of $H_k(\partial\Omega) = H_k(\Omega) \oplus H_k(\bar{\Omega})$.

Now we compute

$$\begin{aligned}\partial\sigma_i &= \partial((-1)^{k+1}[\bar{\Omega}] \cap t_i^*) = (-1)^{2k+1}(\partial[\bar{\Omega}] \cap t_i^* - [\bar{\Omega}] \cap \delta t_i^*) \\ &= (-1)^{2k+1}(-[\partial\Omega] \cap t_i^*) = [\partial\Omega] \cap t_i^*.\end{aligned} \tag{34}$$

We already know that $[\partial\Omega] \cap t_i^* = s_i + d_l t_l$. Since $\partial\sigma_i$ is a linear combination of s_j, we have shown that $\partial\sigma_i = s_i$, as desired.

Notice that we needed the $(-1)^{k+1}$ in our definition of $\sigma_i = (-1)^{k+1}[\bar{\Omega}] \cap t_i^*$ to make the signs come out correctly in this last sequence of arguments. To verify this sign, observe that we can directly compute

$$\begin{aligned}\mathrm{Lk}(t_j, s_i) &= \mathrm{Int}(t_j, \sigma_i) = (t_j^* \cup \sigma_i^*)([\bar{\Omega}]) \\ &= \sigma_i^*([\bar{\Omega}] \cap t_j^*) = (-1)^{k+1}\sigma_i^*(\sigma_j) = (-1)^{k+1}\delta_{ij},\end{aligned}$$

which agrees with our previous computation

$$\mathrm{Lk}(t_j, s_i) = (-1)^{(k+1)^2}\,\mathrm{Lk}(s_i, t_j) = (-1)^{(k+1)^2}\delta_{ij}.$$

Now consider the cup product $s_i^* \cup s_j^*$. Using (31) and (32), we have

$$(s_i^* \cup s_j^*)[\partial\Omega] = s_j^*([\partial\Omega] \cap s_i^*) = (-1)^k s_j^*(t_i) = 0.$$

Similarly, using (31) and (34), we have

$$(t_i^* \cup t_j^*)[\partial\Omega] = t_j^*([\partial\Omega] \cap t_i^*) = t_j^*(s_i) = 0. \qquad \square$$

We now prove our second theorem about the Alexander basis.

Proof of Theorem B.3. Since f is an orientation-preserving homeomorphism, we have $f_*[\partial\Omega] = [\partial\Omega']$. We also know that in $H_k(\Omega')$ and $H_k(\Omega)$, $f^*(s_i'^*) = s_i^*$ since we have $f_*(s_i) = s_i'$. This means that in $H_k(\partial\Omega)$, $\partial f_*(s_i) = s_i' + c_{ji}t_j'$ for some coefficients c_{ji}, since the t_j' span the kernel of the inclusion homomorphism from $H_k(\partial\Omega')$ to $H_k(\Omega')$.

We now compute $f^*(\tau_j'^*) \in H^{k+1}(\Omega, \partial\Omega)$. We know

$$f^*(s_i'^* \cup (\tau_j')^*) = f^*(s_i'^*) \cup f^*((\tau_j')^*) = s_i^* \cup f^*((\tau_j')^*).$$

On the other hand by conclusion (3) of Theorem B.2, we know

$$f^*(s_i'^* \cup (\tau_j')^*) = f^*(\delta_{ij}[\Omega']^*) = \delta_{ij}[\Omega]^*.$$

Thus, using our construction of $\tau_i = [\Omega] \cap s_i$,

$$\delta_{ij} = (s_i^* \cup f^*((\tau_j')^*))[\Omega] = f^*((\tau_j')^*)([\Omega] \cap s_i^*) = f^*((\tau_j')^*)(\tau_i).$$

and we may conclude that $f^*((\tau_j')^*) = \tau_j^*$. Hence $f_*(\tau_j) = \tau_j'$, and $f_*(t_i) = t_i'$, since $t_i = \partial\tau_i$.

This proves that the map $\partial f_*\colon H_k(\partial\Omega) \to H_k(\partial\Omega')$ can be written in the matrix form above. Since $\partial f_*(s_i) = s'_i + c_{ji}t'_j$, it follows that $s^*_i = \partial f^*(s'^*_i + c_{ji}t'^*_j)$. Of course, using conclusion (3) of Theorem B.2 again, we know

$$0 = s^*_i \cup s^*_j = \partial f^*(s'^*_i + c_{ki}t'^*_k) \cup \partial f^*(s'^*_j + c_{jl}t'^*_l).$$

In addition, we know ∂f^* is an isomorphism and $t^*_i \cup s^*_j = \delta_{ij}[\partial\Omega']^*$ so

$$\begin{aligned} 0 &= (s'^*_i + c_{ik}t'^*_k) \cup (s'^*_j + c_{jl}t'^*_l) = c_{jl}s'^*_i \cup t'^*_l + c_{ik}t'^*_k \cup s'^*_j \\ &= (c_{jl}(-1)^{k^2}\delta_{il} + c_{ik}\delta_{kj})[\partial\Omega']^* = ((-1)^{k^2}c_{ji} + c_{ij})[\partial\Omega']^*. \end{aligned}$$

If k is even, this equation becomes $c_{ji} + c_{ij} = 0$ and the matrix is skew-symmetric, while if k is odd, this equation becomes $-c_{ji} + c_{ij} = 0$ and the matrix is symmetric, as claimed. □

It is tempting to wonder what happens if one computes the cup products $s^*_i \cup t^*_j = \partial f^*(s'^*_i + c_{ki}t'^*_k) \cup \partial f^*(t'_j)$ or $t^*_i \cup t^*_j = \partial f^*(t'^*_i) \cup \partial f^*(t'^*_j)$. It is certainly possible to do so, yielding expansions similar to those above, but it turns out to be the case that this procedure yields no additional information about the c_{ij}. Thus we believe that Theorems B.2 and B.3 summarize all of the cohomological information available for an arbitrary compact domain with boundary in $\mathbb{R}^{2k+1}$.

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