Fiber Bundles

A 'short exact sequence of spaces' $A \hookrightarrow X \to X/A$ gives rise to a long exact sequence of homology groups, but not to a long exact sequence of homotopy groups due to the failure of excision. However, there is a different sort of 'short exact sequence of spaces' that does give a long exact sequence of homotopy groups. This sort of short exact sequence $F \longrightarrow E \xrightarrow{p} B$, called a *fiber bundle*, is distinguished from the type $A \hookrightarrow X \to X/A$ in that it has more homogeneity: All the subspaces $p^{-1}(b) \subset E$, which are called **fibers**, are homeomorphic. For example, E could be the product $F \times B$ with $p : E \to B$ the projection. General fiber bundles can be thought of as twisted products. Familiar examples are the Möbius band, which is a twisted annulus with line segments as fibers, and the Klein bottle, which is a twisted torus with circles as fibers.

The topological homogeneity of all the fibers of a fiber bundle is rather like the algebraic homogeneity in a short exact sequence of groups $0 \to K \longrightarrow G \xrightarrow{p} H \to 0$ where the 'fibers' $p^{-1}(h)$ are the cosets of K in G. In a few fiber bundles $F \to E \to B$ the space E is actually a group, F is a subgroup (though seldom a normal subgroup), and B is the space of left or right cosets. One of the nicest such examples is the Hopf bundle $S^1 \to S^3 \to S^2$ where S^3 is the group of quaternions of unit norm and S^1 is the subgroup of unit complex numbers. For this bundle, the long exact sequence of homotopy groups takes the form

$$\cdots \longrightarrow \pi_i(S^1) \longrightarrow \pi_i(S^3) \longrightarrow \pi_i(S^2) \longrightarrow \pi_{i-1}(S^1) \longrightarrow \pi_{i-1}(S^3) \longrightarrow \cdots$$

In particular, the exact sequence gives an isomorphism $\pi_2(S^2) \approx \pi_1(S^1)$ since the two adjacent terms $\pi_2(S^3)$ and $\pi_1(S^3)$ are zero by cellular approximation. Thus we have a direct homotopy-theoretic proof that $\pi_2(S^2) \approx \mathbb{Z}$. Also, since $\pi_i(S^1) = 0$ for $i > 1$ by Proposition 4.1, the exact sequence implies that there are isomorphisms $\pi_i(S^3) \approx \pi_i(S^2)$ for all $i \ge 3$, so in particular $\pi_3(S^2) \approx \pi_3(S^3)$, and by Corollary 4.25 the latter group is $\mathbb{Z}$.

After these preliminary remarks, let us begin by defining the property that leads to a long exact sequence of homotopy groups. A map $p : E \to B$ is said to have the **homotopy lifting property** with respect to a space X if, given a homotopy $g_t : X \to B$ and a map $\widetilde{g}_0 : X \to E$ lifting g_0, so $p\widetilde{g}_0 = g_0$, then there exists a homotopy $\widetilde{g}_t : X \to E$ lifting g_t. From a formal point of view, this can be regarded as a special case of the *lift extension property for a pair* (Z, A), which asserts that every map $Z \to B$ has a lift $Z \to E$ extending a given lift defined on the subspace $A \subset Z$. The case $(Z, A) = (X \times I, X \times \{0\})$ is the homotopy lifting property.

A **fibration** is a map $p : E \to B$ having the homotopy lifting property with respect to all spaces X. For example, a projection $B \times F \to B$ is a fibration since we can choose lifts of the form $\widetilde{g}_t(x) = (g_t(x), h(x))$ where $\widetilde{g}_0(x) = (g_0(x), h(x))$.

Theorem 4.41. *Suppose $p: E \to B$ has the homotopy lifting property with respect to disks D^k for all $k \geq 0$. Choose basepoints $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the map $p_*: \pi_n(E, F, x_0) \to \pi_n(B, b_0)$ is an isomorphism for all $n \geq 1$. Hence if B is path-connected, there is a long exact sequence*

$$\cdots \to \pi_n(F, x_0) \to \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B, b_0) \to \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(E, x_0) \to 0$$

The proof will use a relative form of the homotopy lifting property. The map $p: E \to B$ is said to have the **homotopy lifting property for a pair** (X, A) if each homotopy $f_t: X \to B$ lifts to a homotopy $\widetilde{g}_t: X \to E$ starting with a given lift $\widetilde{g}_0$ and extending a given lift $\widetilde{g}_t: A \to E$. In other words, the homotopy lifting property for (X, A) is the lift extension property for $(X \times I, X \times \{0\} \cup A \times I)$.

The homotopy lifting property for D^k is equivalent to the homotopy lifting property for $(D^k, \partial D^k)$ since the pairs $(D^k \times I, D^k \times \{0\})$ and $(D^k \times I, D^k \times \{0\} \cup \partial D^k \times I)$ are homeomorphic. This implies that the homotopy lifting property for disks is equivalent to the homotopy lifting property for all CW pairs (X, A). For by induction over the skeleta of X it suffices to construct a lifting $\widetilde{g}_t$ one cell of $X - A$ at a time. Composing with the characteristic map $\Phi: D^k \to X$ of a cell then gives a reduction to the case $(X, A) = (D^k, \partial D^k)$. A map $p: E \to B$ satisfying the homotopy lifting property for disks is sometimes called a *Serre fibration*.

Proof: First we show that p_* is onto. Represent an element of $\pi_n(B, b_0)$ by a map $f: (I^n, \partial I^n) \to (B, b_0)$. The constant map to x_0 provides a lift of f to E over the subspace $J^{n-1} \subset I^n$, so the relative homotopy lifting property for $(I^{n-1}, \partial I^{n-1})$ extends this to a lift $\widetilde{f}: I^n \to E$, and this lift satisfies $\widetilde{f}(\partial I^n) \subset F$ since $f(\partial I^n) = b_0$. Then $\widetilde{f}$ represents an element of $\pi_n(E, F, x_0)$ with $p_*([\widetilde{f}]) = [f]$ since $p\widetilde{f} = f$.

Injectivity of p_* is similar. Given $\widetilde{f}_0, \widetilde{f}_1: (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ such that $p_*([\widetilde{f}_0]) = p_*([\widetilde{f}_1])$, let $G: (I^n \times I, \partial I^n \times I) \to (B, b_0)$ be a homotopy from $p\widetilde{f}_0$ to $p\widetilde{f}_1$. We have a partial lift $\widetilde{G}$ given by $\widetilde{f}_0$ on $I^n \times \{0\}$, $\widetilde{f}_1$ on $I^n \times \{1\}$, and the constant map to x_0 on $J^{n-1} \times I$. After permuting the last two coordinates of $I^n \times I$, the relative homotopy lifting property gives an extension of this partial lift to a full lift $\widetilde{G}: I^n \times I \to E$. This is a homotopy $\widetilde{f}_t: (I^n, \partial I^n, J^{n-1}) \to (E, F, x_0)$ from $\widetilde{f}_0$ to $\widetilde{f}_1$. So p_* is injective.

For the last statement of the theorem we plug $\pi_n(B, b_0)$ in for $\pi_n(E, F, x_0)$ in the long exact sequence for the pair (E, F). The map $\pi_n(E, x_0) \to \pi_n(E, F, x_0)$ in the exact sequence then becomes the composition $\pi_n(E, x_0) \to \pi_n(E, F, x_0) \xrightarrow{p_*} \pi_n(B, b_0)$, which is just $p_*: \pi_n(E, x_0) \to \pi_n(B, b_0)$. The 0 at the end of the sequence, surjectivity of $\pi_0(F, x_0) \to \pi_0(E, x_0)$, comes from the hypothesis that B is path-connected since a path in E from an arbitrary point $x \in E$ to F can be obtained by lifting a path in B from $p(x)$ to b_0. $\square$

A **fiber bundle** structure on a space E, with fiber F, consists of a projection map $p: E \to B$ such that each point of B has a neighborhood U for which there is a homeomorphism $h:p^{-1}(U)\to U\times F$ making the diagram at the right commute, where the unlabeled map is projection onto the first factor. Commutativity of the diagram means that h carries each fiber $F_b = p^{-1}(b)$ homeomorphically onto the copy $\{b\}\times F$ of F. Thus the fibers F_b are arranged locally as in the product $B\times F$, though not necessarily globally. An h as above is called a **local trivialization** of the bundle. Since the first coordinate of h is just p, h is determined by its second coordinate, a map $p^{-1}(U)\to F$ which is a homeomorphism on each fiber F_b.

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U\times F \arrow[dl] \\ & U & \end{tikzcd}$$

The fiber bundle structure is determined by the projection map $p:E\to B$, but to indicate what the fiber is we sometimes write a fiber bundle as $F\to E\to B$, a 'short exact sequence of spaces.' The space B is called the **base space** of the bundle, and E is the **total space**.

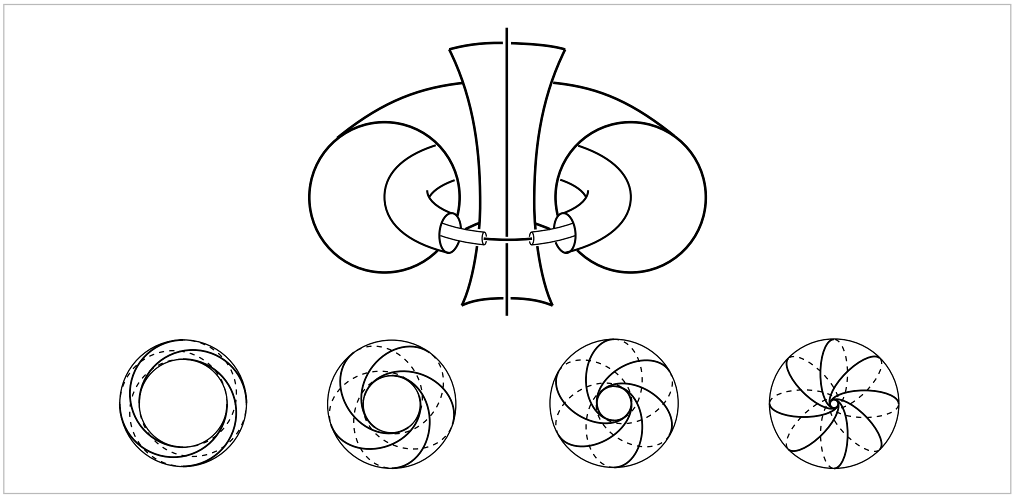
Example 4.42. A fiber bundle with fiber a discrete space is a covering space. Conversely, a covering space whose fibers all have the same cardinality, for example a covering space over a connected base space, is a fiber bundle with discrete fiber.

Example 4.43. One of the simplest nontrivial fiber bundles is the Möbius band, which is a bundle over S^1 with fiber an interval. Specifically, take E to be the quotient of $I\times[-1,1]$ under the identifications $(0,v)\sim(1,-v)$, with $p:E\to S^1$ induced by the projection $I\times[-1,1]\to I$, so the fiber is $[-1,1]$. Glueing two copies of E together by the identity map between their boundary circles produces a Klein bottle, a bundle over S^1 with fiber S^1.

Example 4.44. Projective spaces yield interesting fiber bundles. In the real case we have the familiar covering spaces $S^n\to\mathbb{R}\mathrm{P}^n$ with fiber S^0. Over the complex numbers the analog of this is a fiber bundle $S^1\to S^{2n+1}\to\mathbb{C}\mathrm{P}^n$. Here S^{2n+1} is the unit sphere in $\mathbb{C}^{n+1}$ and $\mathbb{C}\mathrm{P}^n$ is viewed as the quotient space of S^{2n+1} under the equivalence relation $(z_0,\cdots,z_n)\sim\lambda(z_0,\cdots,z_n)$ for $\lambda\in S^1$, the unit circle in $\mathbb{C}$. The projection $p:S^{2n+1}\to\mathbb{C}\mathrm{P}^n$ sends $(z_0,\cdots,z_n)$ to its equivalence class $[z_0,\cdots,z_n]$, so the fibers are copies of S^1. To see that the local triviality condition for fiber bundles is satisfied, let $U_i\subset\mathbb{C}\mathrm{P}^n$ be the open set of equivalence classes $[z_0,\cdots,z_n]$ with $z_i\neq 0$. Define $h_i:p^{-1}(U_i)\to U_i\times S^1$ by $h_i(z_0,\cdots,z_n)=([z_0,\cdots,z_n],z_i/|z_i|)$. This takes fibers to fibers, and is a homeomorphism since its inverse is the map $([z_0,\cdots,z_n],\lambda)\mapsto\lambda|z_i|z_i^{-1}(z_0,\cdots,z_n)$, as one checks by calculation.

The construction of the bundle $S^1\to S^{2n+1}\to\mathbb{C}\mathrm{P}^n$ also works when $n=\infty$, so there is a fiber bundle $S^1\to S^\infty\to\mathbb{C}\mathrm{P}^\infty$.

Example 4.45. The case $n=1$ is particularly interesting since $\mathbb{C}\mathrm{P}^1=S^2$ and the bundle becomes $S^1\to S^3\to S^2$ with fiber, total space, and base all spheres. This is known as the **Hopf bundle**, and is of low enough dimension to be seen explicitly. The projection $S^3\to S^2$ can be taken to be $(z_0,z_1)\mapsto z_0/z_1\in\mathbb{C}\cup\{\infty\}=S^2$. In polar coordinates we have $p(r_0e^{i\theta_0},r_1e^{i\theta_1})=(r_0/r_1)e^{i(\theta_0-\theta_1)}$ where $r_0^2+r_1^2=1$. For a fixed ratio $\rho = r_0/r_1 \in (0,\infty)$ the angles θ_0 and θ_1 vary independently over S^1, so the points $(r_0e^{i\theta_0}, r_1e^{i\theta_1})$ form a torus $T_\rho \subset S^3$. Letting ρ vary, these disjoint tori T_ρ fill up S^3, if we include the limiting cases T_0 and T_∞ where the radii r_0 and r_1 are zero, making the tori T_0 and T_∞ degenerate to circles. These two circles are the unit circles in the two $\mathbb{C}$ factors of $\mathbb{C}^2$, so under stereographic projection of S^3 from the point $(0,1)$ onto $\mathbb{R}^3$ they correspond to the unit circle in the xy-plane and the z-axis. The concentric tori T_ρ are then arranged as in the following figure.



Each torus T_ρ is a union of circle fibers, the pairs (θ_0, θ_1) with $\theta_0 - \theta_1$ constant. These fiber circles have slope 1 on the torus, winding around once longitudinally and once meridionally. With respect to the ambient space it might be more accurate to say they have slope ρ. As ρ goes to 0 or ∞ the fiber circles approach the circles T_0 and T_∞, which are also fibers. The figure shows four of the tori decomposed into fibers.

Example 4.46. Replacing the field $\mathbb{C}$ by the quaternions $\mathbb{H}$, the same constructions yield fiber bundles $S^3 \to S^{4n+3} \to \mathbb{H}\mathrm{P}^n$ over quaternionic projective spaces $\mathbb{H}\mathrm{P}^n$. Here the fiber S^3 is the unit quaternions, and S^{4n+3} is the unit sphere in $\mathbb{H}^{n+1}$. Taking $n = 1$ gives a second Hopf bundle $S^3 \to S^7 \to S^4 = \mathbb{H}\mathrm{P}^1$.

Example 4.47. Another Hopf bundle $S^7 \to S^{15} \to S^8$ can be defined using the octonion algebra $\mathbb{O}$. Elements of $\mathbb{O}$ are pairs of quaternions (a_1, a_2) with multiplication given by $(a_1,a_2)(b_1,b_2) = (a_1b_1 - \overline{b}_2a_2, a_2\overline{b}_1 + b_2a_1)$. Regarding S^{15} as the unit sphere in the 16-dimensional vector space $\mathbb{O}^2$, the projection map $p: S^{15} \to S^8 = \mathbb{O} \cup \{\infty\}$ is $(z_0, z_1) \mapsto z_0z_1^{-1}$, just as for the other Hopf bundles, but because $\mathbb{O}$ is not associative, a little care is needed to show this is a fiber bundle with fiber S^7, the unit octonions. Let U_0 and U_1 be the complements of ∞ and 0 in the base space $\mathbb{O} \cup \{\infty\}$. Define $h_i : p^{-1}(U_i) \to U_i \times S^7$ and $g_i : U_i \times S^7 \to p^{-1}(U_i)$ by

$$h_0(z_0,z_1) = (z_0z_1^{-1}, z_1/|z_1|), \qquad g_0(z,w) = (zw,w)/|(zw,w)|$$
$$h_1(z_0,z_1) = (z_0z_1^{-1}, z_0/|z_0|), \qquad g_1(z,w) = (w,z^{-1}w)/|(w,z^{-1}w)|$$

If one assumes the known fact that any subalgebra of $\mathbb{O}$ generated by two elements is associative, then it is a simple matter to check that g_i and h_i are inverse homeomorphisms, so we have a fiber bundle $S^7 \to S^{15} \to S^8$. Actually, the calculation that g_i and h_i are inverses needs only the following more elementary facts about octonions z, w, where the conjugate $\overline{z}$ of $z = (a_1, a_2)$ is defined by the expected formula $\overline{z} = (\overline{a}_1, -a_2)$:

(1) $rz = zr$ for all $r \in \mathbb{R}$ and $z \in \mathbb{O}$, where $\mathbb{R} \subset \mathbb{O}$ as the pairs $(r, 0)$.
(2) $|z|^2 = z\overline{z} = \overline{z}z$, hence $z^{-1} = \overline{z}/|z|^2$.
(3) $|zw| = |z||w|$.
(4) $\overline{zw} = \overline{w}\,\overline{z}$, hence $(zw)^{-1} = w^{-1}z^{-1}$.
(5) $z(\overline{z}w) = (z\overline{z})w$ and $(zw)\overline{w} = z(w\overline{w})$, hence $z(z^{-1}w) = w$ and $(zw)w^{-1} = z$.

These facts can be checked by somewhat tedious direct calculation. More elegant derivations can be found in Chapter 8 of [Ebbinghaus 1991].

There is an octonion projective plane $\mathbb{O}\mathrm{P}^2$ obtained by attaching a cell e^{16} to S^8 via the Hopf map $S^{15} \to S^8$, just as $\mathbb{C}\mathrm{P}^2$ and $\mathbb{H}\mathrm{P}^2$ are obtained from the other Hopf maps. However, there is no octonion analog of $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^n$, and $\mathbb{H}\mathrm{P}^n$ for $n > 2$ since associativity of multiplication is needed for the relation $(z_0, \cdots, z_n) \sim \lambda(z_0, \cdots, z_n)$ to be an equivalence relation.

There are no fiber bundles with fiber, total space, and base space spheres of other dimensions than in these Hopf bundle examples. This is discussed in an exercise for §4.D, which reduces the question to the famous 'Hopf invariant one' problem.

Proposition 4.48. *A fiber bundle $p: E \to B$ has the homotopy lifting property with respect to all CW pairs (X, A).*

A theorem of Huebsch and Hurewicz proved in §2.7 of [Spanier 1966] says that fiber bundles over paracompact base spaces are fibrations, having the homotopy lifting property with respect to all spaces. This stronger result is not often needed in algebraic topology, however.

Proof: As noted earlier, the homotopy lifting property for CW pairs is equivalent to the homotopy lifting property for disks, or equivalently, cubes. Let $G: I^n \times I \to B$, $G(x, t) = g_t(x)$, be a homotopy we wish to lift, starting with a given lift $\widetilde{g}_0$ of g_0. Choose an open cover $\{U_\alpha\}$ of B with local trivializations $h_\alpha: p^{-1}(U_\alpha) \to U_\alpha \times F$. Using compactness of $I^n \times I$, we may subdivide I^n into small cubes C and I into intervals $I_j = [t_j, t_{j+1}]$ so that each product $C \times I_j$ is mapped by G into a single U_α. We may assume by induction on n that $\widetilde{g}_t$ has already been constructed over ∂C for each of the subcubes C. To extend this $\widetilde{g}_t$ over a cube C we may proceed in stages, constructing $\widetilde{g}_t$ for t in each successive interval I_j. This in effect reduces us to the case that no subdivision of $I^n \times I$ is necessary, so G maps all of $I^n \times I$ to a single U_α. Then we have $\widetilde{G}(I^n \times \{0\} \cup \partial I^n \times I) \subset p^{-1}(U_\alpha)$, and composing $\widetilde{G}$ with the local trivialization

h_α reduces us to the case of a product bundle $U_\alpha \times F$. In this case the first coordinate of a lift $\widetilde{g}_t$ is just the given g_t, so only the second coordinate needs to be constructed. This can be obtained as a composition $I^n \times I \to I^n \times \{0\} \cup \partial I^n \times I \to F$ where the first map is a retraction and the second map is what we are given. $\square$

Example 4.49. Applying this theorem to a covering space $p: E \to B$ with E and B path-connected, and discrete fiber F, the resulting long exact sequence of homotopy groups yields Proposition 4.1 that $p_*: \pi_n(E) \to \pi_n(B)$ is an isomorphism for $n \ge 2$. We also obtain a short exact sequence $0 \to \pi_1(E) \to \pi_1(B) \to \pi_0(F) \to 0$, consistent with the covering space theory facts that $p_*: \pi_1(E) \to \pi_1(B)$ is injective and that the fiber F can be identified, via path-lifting, with the set of cosets of $p_*\pi_1(E)$ in $\pi_1(B)$.

Example 4.50. From the bundle $S^1 \to S^\infty \to \mathbb{C}\mathrm{P}^\infty$ we obtain $\pi_i(\mathbb{C}\mathrm{P}^\infty) \approx \pi_{i-1}(S^1)$ for all i since S^∞ is contractible. Thus $\mathbb{C}\mathrm{P}^\infty$ is a $K(\mathbb{Z}, 2)$. In similar fashion the bundle $S^3 \to S^\infty \to \mathbb{H}\mathrm{P}^\infty$ gives $\pi_i(\mathbb{H}\mathrm{P}^\infty) \approx \pi_{i-1}(S^3)$ for all i, but these homotopy groups are far more complicated than for $\mathbb{C}\mathrm{P}^\infty$ and S^1. In particular, $\mathbb{H}\mathrm{P}^\infty$ is not a $K(\mathbb{Z}, 4)$.

Example 4.51. The long exact sequence for the Hopf bundle $S^1 \to S^3 \to S^2$ gives isomorphisms $\pi_2(S^2) \approx \pi_1(S^1)$ and $\pi_n(S^3) \approx \pi_n(S^2)$ for all $n \ge 3$. Taking $n = 3$, we see that $\pi_3(S^2)$ is infinite cyclic, generated by the Hopf map $S^3 \to S^2$.

From this example and the preceding one we see that S^2 and $S^3 \times \mathbb{C}\mathrm{P}^\infty$ are simply-connected CW complexes with isomorphic homotopy groups, though they are not homotopy equivalent since they have quite different homology groups.

Example 4.52: Whitehead Products. Let us compute $\pi_3(\bigvee_\alpha S^2_\alpha)$, showing that it is free abelian with basis consisting of the Hopf maps $S^3 \to S^2_\alpha \subset \bigvee_\alpha S^2_\alpha$ together with the attaching maps $S^3 \to S^2_\alpha \vee S^2_\beta \subset \bigvee_\alpha S^2_\alpha$ of the cells $e^2_\alpha \times e^2_\beta$ in the products $S^2_\alpha \times S^2_\beta$ for all unordered pairs $\alpha \ne \beta$.

Suppose first that there are only finitely many summands S^2_α. For a finite product $\prod_\alpha X_\alpha$ of path-connected spaces, the map $\pi_n(\bigvee_\alpha X_\alpha) \to \pi_n(\prod_\alpha X_\alpha)$ induced by inclusion is surjective since the group $\pi_n(\prod_\alpha X_\alpha) \approx \bigoplus_\alpha \pi_n(X_\alpha)$ is generated by the subgroups $\pi_n(X_\alpha)$. Thus the long exact sequence of homotopy groups for the pair $(\prod_\alpha X_\alpha, \bigvee_\alpha X_\alpha)$ breaks up into short exact sequences

$$0 \longrightarrow \pi_{n+1}(\textstyle\prod_\alpha X_\alpha, \bigvee_\alpha X_\alpha) \longrightarrow \pi_n(\bigvee_\alpha X_\alpha) \longrightarrow \pi_n(\prod_\alpha X_\alpha) \longrightarrow 0$$

These short exact sequences split since the inclusions $X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ induce maps $\pi_n(X_\alpha) \to \pi_n(\bigvee_\alpha X_\alpha)$ and hence a splitting homomorphism $\bigoplus_\alpha \pi_n(X_\alpha) \to \pi_n(\bigvee_\alpha X_\alpha)$. Taking $X_\alpha = S^2_\alpha$ and $n = 3$, we get an isomorphism

$$\pi_3(\textstyle\bigvee_\alpha S^2_\alpha) \approx \pi_4(\prod_\alpha S^2_\alpha, \bigvee_\alpha S^2_\alpha) \oplus (\bigoplus_\alpha \pi_3(S^2_\alpha))$$

The factor $\bigoplus_\alpha \pi_3(S^2_\alpha)$ is free with basis the Hopf maps $S^3 \to S^2_\alpha$ by the preceding example. For the other factor we have $\pi_4(\prod_\alpha S^2_\alpha, \bigvee_\alpha S^2_\alpha) \approx \pi_4(\prod_\alpha S^2_\alpha / \bigvee_\alpha S^2_\alpha)$ by Proposition 4.28. The quotient $\prod_\alpha S^2_\alpha / \bigvee_\alpha S^2_\alpha$ has 5-skeleton a wedge of spheres $S^4_{\alpha\beta}$ for $\alpha \ne \beta$, so $\pi_4(\prod_\alpha S^2_\alpha/\bigvee_\alpha S^2_\alpha) \approx \pi_4(\bigvee_{\alpha\beta} S^4_{\alpha\beta})$ is free with basis the inclusions $S^4_{\alpha\beta} \hookrightarrow \bigvee_{\alpha\beta} S^4_{\alpha\beta}$. Hence $\pi_4(\prod_\alpha S^2_\alpha, \bigvee_\alpha S^2_\alpha)$ is free with basis the characteristic maps of the 4-cells $e^2_\alpha \times e^2_\beta$. Via the injection $\partial : \pi_4(\prod_\alpha S^2_\alpha, \bigvee_\alpha S^2_\alpha) \to \pi_3(\bigvee_\alpha S^2_\alpha)$ this means that the attaching maps of the cells $e^2_\alpha \times e^2_\beta$ form a basis for the summand $\operatorname{Im}\partial$ of $\pi_3(\bigvee_\alpha S^2_\alpha)$. This finishes the proof for the case of finitely many summands S^2_α. The case of infinitely many S^2_α's follows immediately since any map $S^3 \to \bigvee_\alpha S^2_\alpha$ has compact image, lying in a finite union of summands, and similarly for any homotopy between such maps.

The maps $S^3 \to S^2_\alpha \vee S^2_\beta$ in this example are expressible in terms of a product in homotopy groups called the **Whitehead product**, defined as follows. Given basepoint-preserving maps $f : S^k \to X$ and $g : S^\ell \to X$, let $[f,g] : S^{k+\ell-1} \to X$ be the composition $S^{k+\ell-1} \longrightarrow S^k \vee S^\ell \xrightarrow{f \vee g} X$ where the first map is the attaching map of the $(k+\ell)$-cell of $S^k \times S^\ell$ with its usual CW structure. Since homotopies of f or g give rise to homotopies of $[f,g]$, we have a well-defined product $\pi_k(X) \times \pi_\ell(X) \to \pi_{k+\ell-1}(X)$. The notation $[f,g]$ is used since for $k = \ell = 1$ this is just the commutator product in $\pi_1(X)$. It is an exercise to show that when $k = 1$ and $\ell > 1$, $[f,g]$ is the difference between g and its image under the π_1-action of f.

In these terms the map $S^3 \to S^2_\alpha \vee S^2_\beta$ in the preceding example is the Whitehead product $[i_\alpha, i_\beta]$ of the two inclusions of S^2 into $S^2_\alpha \vee S^2_\beta$. Another example of a Whitehead product we have encountered previously is $[\mathbb{1}, \mathbb{1}] : S^{2n-1} \to S^n$, which is the attaching map of the $2n$-cell of the space $J(S^n)$ considered in §3.2.

The calculation of $\pi_3(\bigvee_\alpha S^2_\alpha)$ is the first nontrivial case of a more general theorem of Hilton calculating all the homotopy groups of any wedge sum of spheres in terms of homotopy groups of spheres, using Whitehead products. A further generalization by Milnor extends this to wedge sums of suspensions of arbitrary connected CW complexes. See [Whitehead 1978] for an exposition of these results and further information on Whitehead products.

Example 4.53: Stiefel and Grassmann Manifolds. The fiber bundles with total space a sphere and base space a projective space considered above are the cases $n = 1$ of families of fiber bundles in each of the real, complex, and quaternionic cases:

$$\begin{array}{ll} O(n) \longrightarrow V_n(\mathbb{R}^k) \longrightarrow G_n(\mathbb{R}^k) & \qquad O(n) \longrightarrow V_n(\mathbb{R}^\infty) \longrightarrow G_n(\mathbb{R}^\infty) \\ U(n) \longrightarrow V_n(\mathbb{C}^k) \longrightarrow G_n(\mathbb{C}^k) & \qquad U(n) \longrightarrow V_n(\mathbb{C}^\infty) \longrightarrow G_n(\mathbb{C}^\infty) \\ Sp(n) \longrightarrow V_n(\mathbb{H}^k) \longrightarrow G_n(\mathbb{H}^k) & \qquad Sp(n) \longrightarrow V_n(\mathbb{H}^\infty) \longrightarrow G_n(\mathbb{H}^\infty) \end{array}$$

Taking the real case first, the Stiefel manifold $V_n(\mathbb{R}^k)$ is the space of n-frames in $\mathbb{R}^k$, that is, n-tuples of orthonormal vectors in $\mathbb{R}^k$. This is topologized as a subspace of the product of n copies of the unit sphere in $\mathbb{R}^k$. The Grassmann manifold $G_n(\mathbb{R}^k)$ is the space of n-dimensional vector subspaces of $\mathbb{R}^k$. There is a natural surjection $p : V_n(\mathbb{R}^k) \to G_n(\mathbb{R}^k)$ sending an n-frame to the subspace it spans, and $G_n(\mathbb{R}^k)$ is topologized as a quotient space of $V_n(\mathbb{R}^k)$ via this projection. The fibers of the map

p are the spaces of n-frames in a fixed n-plane in $\mathbb{R}^k$ and so are homeomorphic to $V_n(\mathbb{R}^n)$. An n-frame in $\mathbb{R}^n$ is the same as an orthogonal $n\times n$ matrix, regarding the columns of the matrix as an n-frame, so the fiber can also be described as the orthogonal group $O(n)$. There is no difficulty in allowing $k=\infty$ in these definitions, and in fact $V_n(\mathbb{R}^\infty)=\bigcup_k V_n(\mathbb{R}^k)$ and $G_n(\mathbb{R}^\infty)=\bigcup_k G_n(\mathbb{R}^k)$.

The complex and quaternionic Stiefel manifolds and Grassmann manifolds are defined in the same way using the usual Hermitian inner products in $\mathbb{C}^k$ and $\mathbb{H}^k$. The unitary group $U(n)$ consists of $n\times n$ matrices whose columns form orthonormal bases for $\mathbb{C}^n$, and the symplectic group $Sp(n)$ is the quaternionic analog of this.

We should explain why the various projection maps $V_n \to G_n$ are fiber bundles. Let us take the real case for concreteness, though the argument is the same in all cases. If we fix an n-plane $P\in G_n(\mathbb{R}^k)$ and choose an orthonormal basis for P, then we obtain continuously varying orthonormal bases for all n-planes P' in a neighborhood U of P by projecting the basis for P orthogonally onto P' to obtain a nonorthonormal basis for P', then applying the Gram–Schmidt process to this basis to make it orthonormal. The formulas for the Gram–Schmidt process show that it is continuous. Having orthonormal bases for all n-planes in U, we can use these to identify these n-planes with $\mathbb{R}^n$, hence n-frames in these n-planes are identified with n-frames in $\mathbb{R}^n$, and so $p^{-1}(U)$ is identified with $U\times V_n(\mathbb{R}^n)$. This argument works for $k=\infty$ as well as for finite k.

In the case $n=1$ the total spaces V_1 are spheres, which are highly connected, and the same is true in general:

- $V_n(\mathbb{R}^k)$ is $(k-n-1)$-connected.
- $V_n(\mathbb{C}^k)$ is $(2k-2n)$-connected.
- $V_n(\mathbb{H}^k)$ is $(4k-4n+2)$-connected.
- $V_n(\mathbb{R}^\infty)$, $V_n(\mathbb{C}^\infty)$, and $V_n(\mathbb{H}^\infty)$ are contractible.

The first three statements will be proved in the next example. For the last statement the argument is the same in the three cases, so let us consider the real case. Define a homotopy $h_t:\mathbb{R}^\infty\to\mathbb{R}^\infty$ by $h_t(x_1,x_2,\cdots)=(1-t)(x_1,x_2,\cdots)+t(0,x_1,x_2,\cdots)$. This is linear for each t, and its kernel is easily checked to be trivial. So if we apply h_t to an n-frame we get an n-tuple of independent vectors, which can be made orthonormal by the Gram–Schmidt formulas. Thus we have a deformation retraction, in the weak sense, of $V_n(\mathbb{R}^\infty)$ onto the subspace of n-frames with first coordinate zero. Iterating this n times, we deform into the subspace of n-frames with first n coordinates zero. For such an n-frame $(v_1,\cdots,v_n)$ define a homotopy $(1-t)(v_1,\cdots,v_n)+t(e_1,\cdots,e_n)$ where e_i is the i^{th} standard basis vector in $\mathbb{R}^\infty$. This homotopy preserves linear independence, so after again applying Gram–Schmidt we have a deformation through n-frames, which finishes the construction of a contraction of $V_n(\mathbb{R}^\infty)$.

Since $V_n(\mathbb{R}^\infty)$ is contractible, we obtain isomorphisms $\pi_iO(n)\approx\pi_{i+1}G_n(\mathbb{R}^\infty)$ for all i and n, and similarly in the complex and quaternionic cases.

Example 4.54. For $m < n \le k$ there are fiber bundles

$$V_{n-m}(\mathbb{R}^{k-m}) \longrightarrow V_n(\mathbb{R}^k) \xrightarrow{p} V_m(\mathbb{R}^k)$$

where the projection p sends an n-frame onto the m-frame formed by its first m vectors, so the fiber consists of $(n-m)$-frames in the $(k-m)$-plane orthogonal to a given m-frame. Local trivializations can be constructed as follows. For an m-frame F, choose an orthonormal basis for the $(k-m)$-plane orthogonal to F. This determines orthonormal bases for the $(k-m)$-planes orthogonal to all nearby m-frames by orthogonal projection and Gram–Schmidt, as in the preceding example. This allows us to identify these $(k-m)$-planes with $\mathbb{R}^{k-m}$, and in particular the fibers near $p^{-1}(F)$ are identified with $V_{n-m}(\mathbb{R}^{k-m})$, giving a local trivialization.

There are analogous bundles in the complex and quaternionic cases as well, with local triviality shown in the same way.

Restricting to the case $m = 1$, we have bundles $V_{n-1}(\mathbb{R}^{k-1}) \to V_n(\mathbb{R}^k) \to S^{k-1}$ whose associated long exact sequence of homotopy groups allows us deduce that $V_n(\mathbb{R}^k)$ is $(k-n-1)$-connected by induction on n. In the complex and quaternionic cases the same argument yields the other connectivity statements in the preceding example.

Taking $k = n$ we obtain fiber bundles $O(k-m) \to O(k) \to V_m(\mathbb{R}^k)$. The fibers are in fact just the cosets $\alpha O(k-m)$ for $\alpha \in O(k)$, where $O(k-m)$ is regarded as the subgroup of $O(k)$ fixing the first m standard basis vectors. So we see that $V_m(\mathbb{R}^k)$ is identifiable with the coset space $O(k)/O(k-m)$, or in other words the orbit space for the free action of $O(k-m)$ on $O(k)$ by right-multiplication. In similar fashion one can see that $G_m(\mathbb{R}^k)$ is the coset space $O(k)/\big(O(m)\times O(k-m)\big)$ where the subgroup $O(m)\times O(k-m) \subset O(k)$ consists of the orthogonal transformations taking the m-plane spanned by the first m standard basis vectors to itself. The corresponding observations apply also in the complex and quaternionic cases, with the unitary and symplectic groups.

Example 4.55: Bott Periodicity. Specializing the preceding example by taking $m = 1$ and $k = n$ we obtain bundles

$$\begin{aligned} O(n-1) &\longrightarrow O(n) \xrightarrow{p} S^{n-1} \\ U(n-1) &\longrightarrow U(n) \xrightarrow{p} S^{2n-1} \\ Sp(n-1) &\longrightarrow Sp(n) \xrightarrow{p} S^{4n-1} \end{aligned}$$

The map p can be described as evaluation of an orthogonal, unitary, or symplectic transformation on a fixed unit vector. These bundles show that computing homotopy groups of $O(n)$, $U(n)$, and $Sp(n)$ should be at least as difficult as computing homotopy groups of spheres. For example, if one knew the homotopy groups of $O(n)$ and $O(n-1)$, then from the long exact sequence of homotopy groups for the first bundle one could say quite a bit about the homotopy groups of S^{n-1}.

The bundles above imply a very interesting stability property. In the real case, the inclusion $O(n-1) \hookrightarrow O(n)$ induces an isomorphism on π_i for $i < n-2$, from the long exact sequence of the first bundle. Hence the groups $\pi_i O(n)$ are independent of n if n is sufficiently large, and the same is true for the groups $\pi_i U(n)$ and $\pi_i Sp(n)$ via the other two bundles. One of the most surprising results in all of algebraic topology is the Bott Periodicity Theorem which asserts that these stable groups repeat periodically, with a period of eight for O and Sp and a period of two for U. Their values are given in the following table:

i mod 8	0	1	2	3	4	5	6	7
$\pi_i O(n)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
$\pi_i U(n)$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
$\pi_i Sp(n)$	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$

Stable Homotopy Groups

We showed in Corollary 4.24 that for an n-connected CW complex X, the suspension map $\pi_i(X) \to \pi_{i+1}(SX)$ is an isomorphism for $i < 2n+1$. In particular this holds for $i \le n$ so SX is $(n+1)$-connected. This implies that in the sequence of iterated suspensions

$$\pi_i(X) \longrightarrow \pi_{i+1}(SX) \longrightarrow \pi_{i+2}(S^2X) \longrightarrow \cdots$$

all maps are eventually isomorphisms, even without any connectivity assumption on X itself. The resulting **stable homotopy group** is denoted $\pi_i^s(X)$.

An especially interesting case is the group $\pi_i^s(S^0)$, which equals $\pi_{i+n}(S^n)$ for $n > i+1$. This stable homotopy group is often abbreviated to π_i^s and called the **stable *i*-stem**. It is a theorem of Serre which we prove in [SSAT] that π_i^s is always finite for $i > 0$.

These stable homotopy groups of spheres are among the most fundamental objects in topology, and much effort has gone into their calculation. At the present time, complete calculations are known only for i up to around 60 or so. Here is a table for $i \le 19$, taken from [Toda 1962]:

i	0	1	2	3	4	5	6	7	8	9	10	11	12
π_i^s	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_{240}$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_6$	$\mathbb{Z}_{504}$	0

13	14	15	16	17	18	19
$\mathbb{Z}_3$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_{480}\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_8\times\mathbb{Z}_2$	$\mathbb{Z}_{264}\times\mathbb{Z}_2$

Patterns in this apparent chaos begin to emerge only when one projects π_i^s onto its ***p*-components**, the quotient groups obtained by factoring out all elements of order relatively prime to the prime p. For $i > 0$ the p-component ${}_p\pi_i^s$ is of course isomorphic to the subgroup of π_i^s consisting of elements of order a power of p, but the quotient viewpoint is in some ways preferable.