Introduction to Geometric Knot Theory 2: Ropelength and Tight Knots

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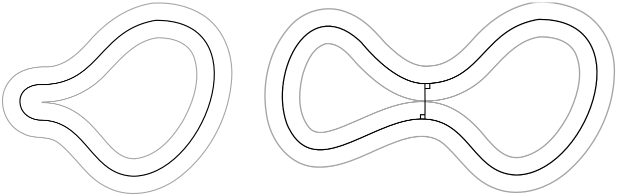
Review from first lecture:

Definition (Federer 1959)

The *reach* of a space curve is the largest ϵ so that any point in an ϵ-neighborhood of the curve has a unique nearest neighbor on the curve.



Idea

reach(K) *(also called thickness) is controlled by curvature maxima (kinks) and self-distance minima (struts).*

Ropelength

Definition

The ropelength of K is given by $\mathrm{Rop}(K) = \mathrm{Len}(K)/\,\mathrm{reach}(K)$.

Theorem (with Kusner, Sullivan 2002, Gonzalez, De la Llave 2003, Gonzalez, Maddocks, Schuricht, Von der Mosel 2002)

Ropelength minimizers (called tight knots) exist in each knot and link type and are $C^{1,1}$.

We can bound Rop in terms of Cr.

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We can bound Rop in terms of Cr. For small knots, the most effective bound is

Theorem (Diao 2006)

$$\mathrm{Rop}(K) \geq \frac{1}{2}\left(17.334 + \sqrt{17.334^2 + 64\pi\,\mathrm{Cr}(K)}\right).$$

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Ropelength minimizers (called tight knots) exist in each knot and link type and are $C^{1,1}$.

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Theorem (Buck and Simon 1999)

$$\operatorname{Rop}(K) \geq 2.210 \operatorname{Cr}^{3/4}.$$

Bounding ropelength in terms of topological invariants

Definition

Peri(n) is the minimum length of any curve surrounding n disjoint unit disks in the plane.

Theorem (with Kusner, Sullivan 2002)

Suppose K is topologically linked to n components and K and all the other components have unit reach. Then

$$\text{Rop}(K) \geq 2\pi + \text{Peri}(n).$$

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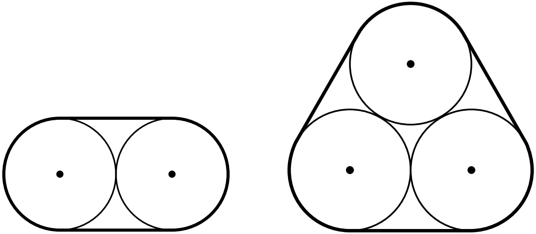

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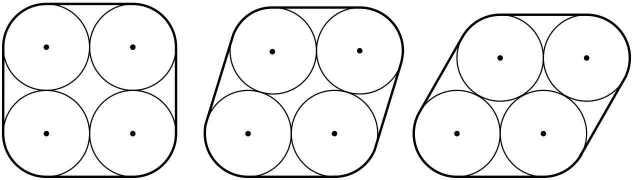

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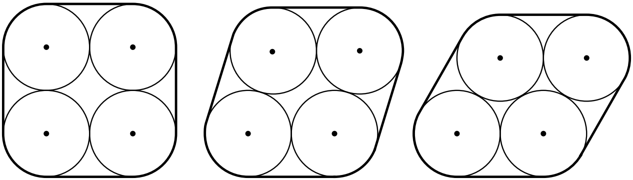

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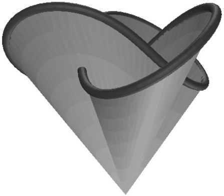
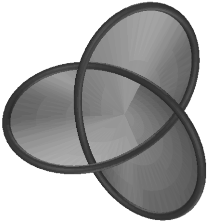
Proof (sketch) of Peri(n) bound for ropelength

Proposition

For any closed curve K of unit reach, there is a point p outside the tube around K so that the cone of K to p has (intrinsic) cone angle 2π.





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The intrinsic geometry of the cone is Euclidean and the other components puncture it in n disjoint disks.

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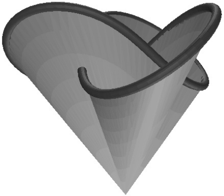

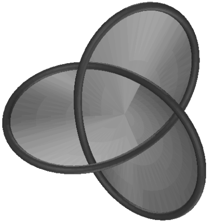

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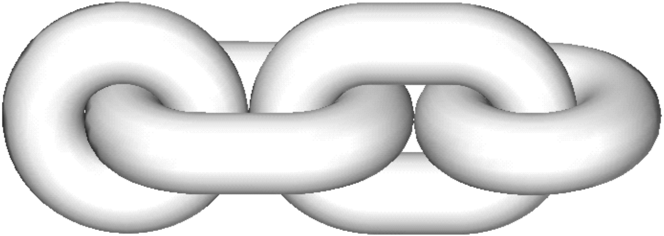
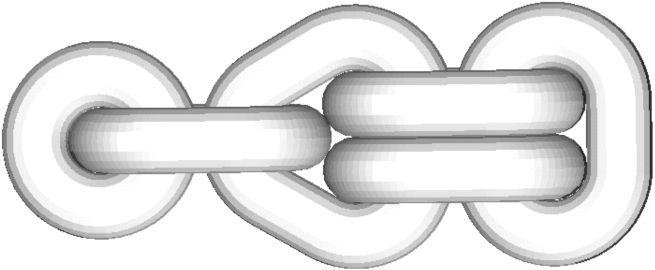
This bound is sometimes sharp

For some examples, the Peri(n) bound is actually sharp.





Linking number bounds for ropelength

Theorem (with Kusner, Sullivan 2002)

If K and J have the same reach, then

$$\mathrm{Rop}(K) \geq 2\pi + 2\pi\sqrt{\mathrm{Lk}(K,J)}.$$

Proof.

A unit norm vector field flowing along the tube around J has flux across the Euclidean cone spanning K of $\pi\,\mathrm{Lk}(K,J)$, so the cone has at least this area. □

Remark

The extra 2π comes from the portion of the spanning disk in the tube around K and depends on cone angle. If K was knotted, we could find a 4π cone point and improve it to 4π.

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Another linking number bound on ropelength

It is interesting to compare this bound to

Theorem (Diao, Janse Van Rensburg 2002)

If K and J have unit reach, is a constant c_2 so that

$$\min\{\operatorname{Len}(K)\operatorname{Len}(J)^{1/3}, \operatorname{Len}(K)^{1/3}\operatorname{Len}(J)\} \geq c_2 \operatorname{Lk}(K, J)$$

Proof.

Proved by directly bounding the Gauss linking integral

$$\operatorname{Lk}(K, J) = \frac{1}{4\pi}\iint \frac{K'(s) \times J'(t) \cdot (K(s) - J(t))}{|K(s) - J(t)|^3}\, \mathrm{d}s\, \mathrm{d}t.$$

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Open question

Open Question

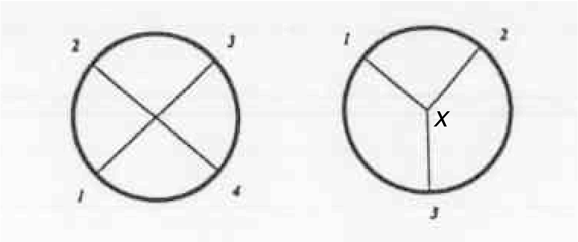
Can you find bounds on ropelength in terms of finite-type invariants by looking at their integral formulations?

Definition

Let ω be the pullback of area form on S^2 to $\mathbb{R}^3$ under $x \mapsto x/|x|$.

For example, we note that the Gauss integral can be written

$$\operatorname{Lk}(K, J) = \int_{S^1 \times S^1} \omega(K(s) - J(t)).$$



Definition

Let $\Delta_4 = \{(s_1, s_2, s_3, s_4) | s_1, s_2, s_3, s_4$ in order on $S^1\}$ and

$$\Delta_3 = \{(s_1, s_2, s_3; x) | s_1, s_2, s_3 \text{ in order on } S^1 \text{ and } x \in \mathbb{R}^3 \text{ not on } K(S^1)\}.$$

Open question (continued)

Theorem (Guadagnini, Martinelli, Minchev 1989, Bar-Natan 1991, cf. Bott, Taubes 1995, Lin, Wang 1996)

The second coefficient of the Conway polynomial v_2 (normalized so $v_2(\text{unknot}) = -1/24$) obeys

$$v_2 = \int_{\Delta_4} \omega(K(s_3) - K(s_1)) \wedge \omega(K(s_4) - K(s_2)) - \int_{\Delta_3} \omega(x - K(s_1)) \wedge \omega(x - K(s_2)) \wedge \omega(x - K(s_3))$$

Open Question

In particular, can you bound this integral for v_2 above in terms of ropelength?

Other finite-type invariants

Theorem (Thurston 1995, Altschuler and Friedel 1995)

All of the finite type invariants have integral formulations defined in terms of linear combinations of Gauss-type integrals of configuration spaces of points on the knots and in space.

(Actually defining the integrals would take too long to do here.)

Open Question

Is ropelength bounded below by a certain power of any finite-type invariant of type n? If so, what power?

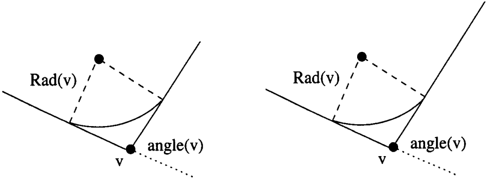
Approximating Ropelength Minimizers

Definition

The ropelength of a polygon is defined by

$$\text{Rop}(P) = \min\left\{\text{MinRad}(P), \frac{\text{dcsd}\, P}{2}\right\}.$$

where MinRad(P) is the minimum radius of all the circle arcs inscribed at vertices of P so that they are tangent to P at both ends and touch the midpoint of the shorter edge at each vertex.



A hunting license

Theorem (Rawdon 2000)

Suppose that P is a polygonal knot. Then there exists a $C^{1,1}$ knot K inscribed in P so that

$$\operatorname{Rop}(P) \geq \operatorname{Rop}(K)$$

Given this theorem, we can use computational methods to find upper bounds for smooth ropelength by finding tight polygonal knots.

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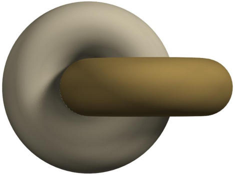

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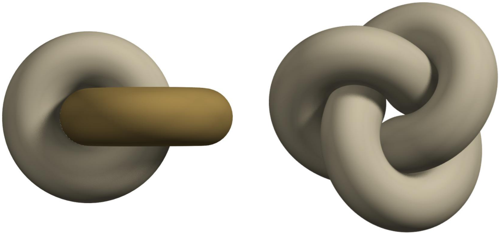

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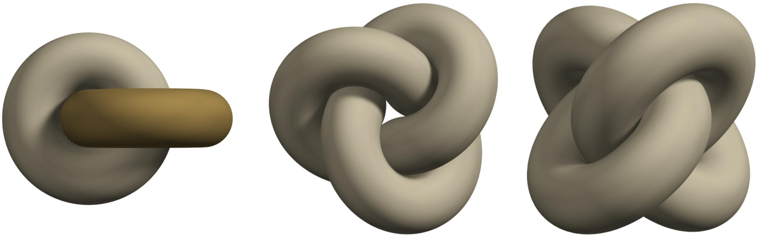

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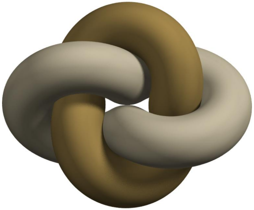

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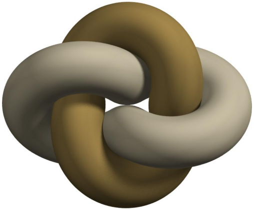

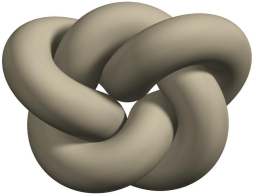

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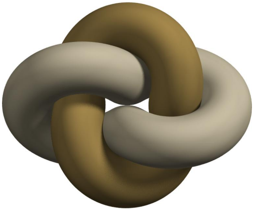
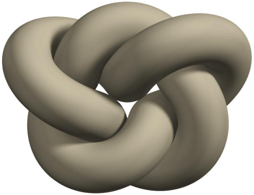
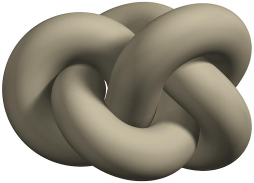

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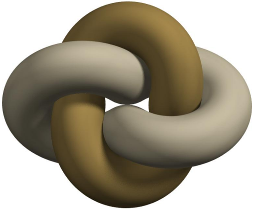

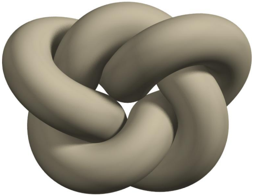

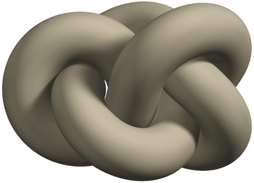

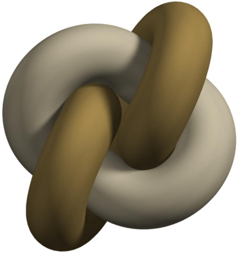

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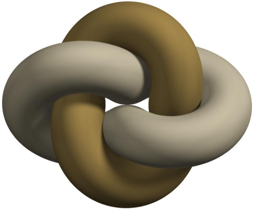
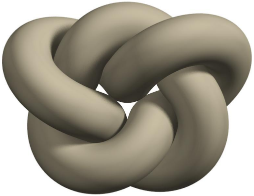
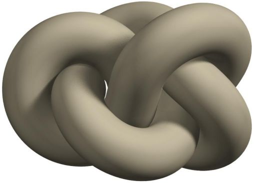
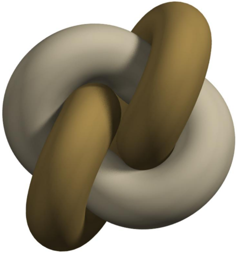
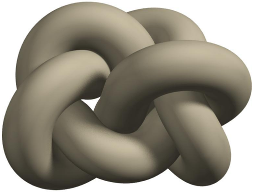

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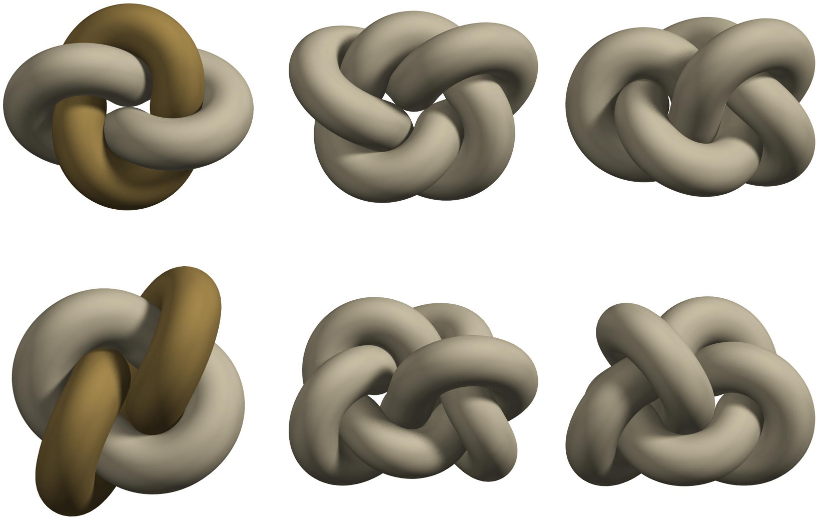

EE

Tightening knots by computer

- Simulated annealing
  - Laurie, Stasiak, et. al. 1997
  - Rawdon 2000-2006 `TOROS`
  - Smutny, Maddocks 2003-2004 (for a kind of spline)
- Gradient Descent
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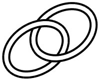
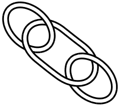
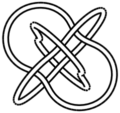
Results (with Ashton, Piatek, Rawdon 2006) `ridgerunner`:

			
Link name	Hopf link (2^2_1)	$2^2_1 \# 2^2_1$	Borromean rings (6^3_2)
Vertices	216	384	630
Rop bound	25.1389	41.7086588	58.0146
Rop	8π	$12\pi + 4$	58.0060
Error	0.02%	0.02%	0.01%

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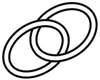
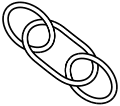
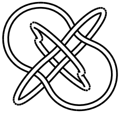

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	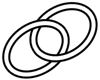	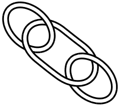	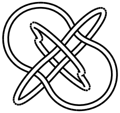
Link name	Hopf link (2^2_1)	$2^2_1 \# 2^2_1$	Borromean rings (6^3_2)
Vertices	216	384	630
Rop bound	25.1389	41.7086588	58.0146
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	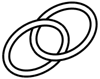	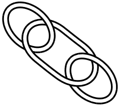	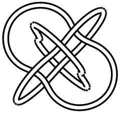
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	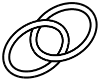	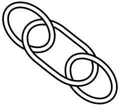	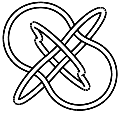
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	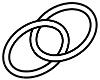	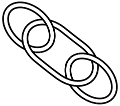	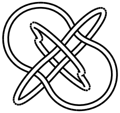
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	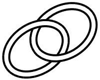	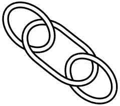	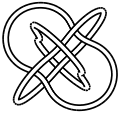
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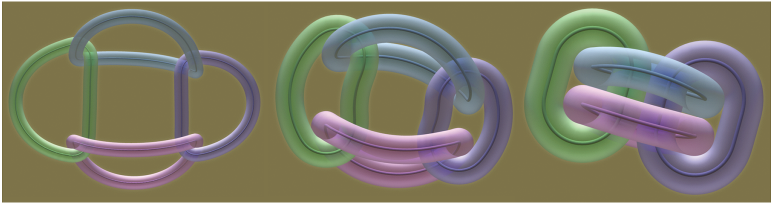
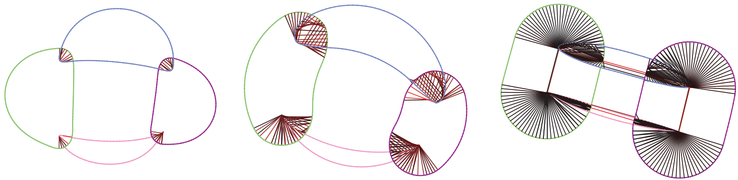
How does it work?

Simulates the gradient flow of length



. . . with struts entered as new constraints as they form . . .



. . . eventually all motion is stopped by constraints.

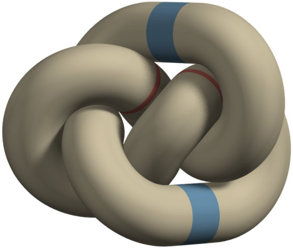
Movies

Nobody could resist showing a few minutes of movie footage from this process. (It's Friday afternoon, after all!)

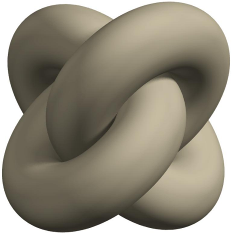
Ok, now back to work . . .

Open Question

Can a tightening knot get "stuck" in a local ropelength minimum before reaching the global minimum?



$4_1\beta$, $\mathrm{Rop}(K) = 44.868$



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Is there an unknotted local minimum for ropelength other than the circle?

Theorem (Smale Conjecture, Hatcher 1983)

The space of smoothly embedded unknotted circles in S^3 deformation retracts onto the space of great circles in S^3.

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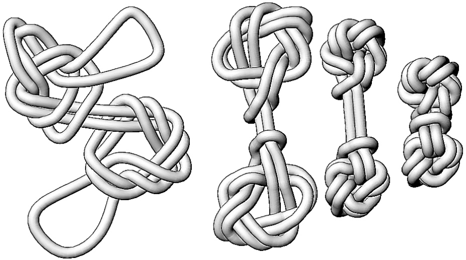

Important open questions

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Find an energy functional for which there is only one unknotted local minimum for energy.

Remark

Of course, this is probably very hard, since it would provide an alternate proof of the Smale conjecture. Freedman tried it in the 1990s without success.

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Ropelength-critical configurations

Definition

The set Kink is the set of two-jets (x, v, a) with radius of curvature 1 in the closure of the set of 2-jets of L. If L is (piecewise) C^2, then Kink is the set of points with radius of curvature λ.

Definition

The set Strut is the set of pairs of points (x, y) on L with $xy \perp L$ at x and y and $|x - y| = 2\,\mathrm{reach}(L)$.

Idea

The struts and kinks prevent L from reducing length without also reducing reach.

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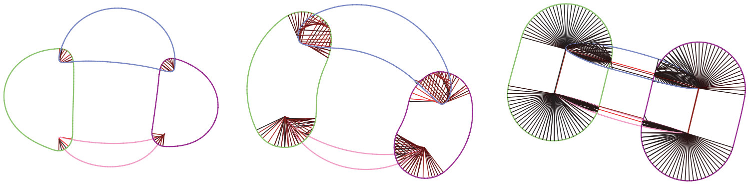
Strut measures



Definition

A *strut measure* is a non-negative Radon measure on the struts representing a compression force pointing outwards.

Definition

A strut force measure $\mathbb{S}$ on L is the vector-valued Radon measure defined at each point p of L by integrating a strut measure over all the struts with an endpoint at p.

Main Theorem

Theorem (with Fu, Kusner, Sullivan, Wrinkle (in preparation))

Suppose L is ropelength-critical, and that Kink *is included in a finite union of closed C^2 subarcs of L. Then* $\exists$ *a strut force measure* $\mathcal{S}$ *and a lower semicontinuous function* $\varphi \in \mathrm{BV}(L)$ *such that* $(\varphi N)' \in \mathrm{BV}(L)$*, with*

$$\mathcal{S}\big|_{\text{interior } L} = -\left((1 - 2\varphi)T - (\varphi N)'\right)'\Big|_{\text{interior } L}.$$

If p is a fixed endpoint of L, $\varphi(p) = 0$.

We are supposed to think of φ as a "kink force measure".

Ideas from the proof

Theorem (an application of ∞-dim'l Kuhn-Tucker theorem)

Suppose L is regular and $\text{reach}(L) \geq 1$*. Then L is ropelength-critical iff there exist nonnegative Radon measures* μ *on* $\text{Strut}(L)$ *and* ν *on* $\text{Kink}(L)$ *such that for any compatible vector field* ξ*,*

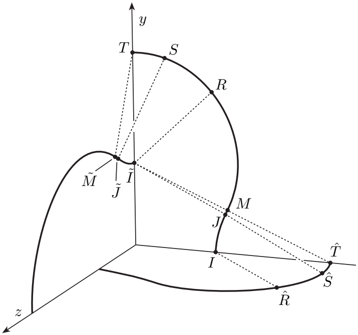
$$\delta_\xi \, \text{length}(L) = \int_{\text{Strut}(L)} \langle x - y, \xi_x - \xi_y \rangle \, d\mu(x, y) + \int_{\text{Kink}(L)} \delta_\xi r \, d\nu(x, v, a).$$

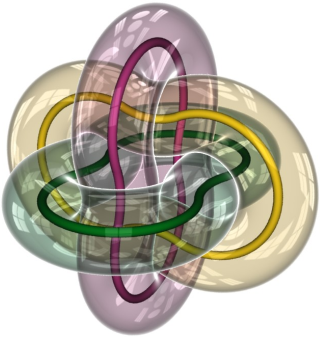
Integrate by parts to derive the Euler-Lagrange equation:

$$\underbrace{S}_{\text{from } d\mu} = -(\underbrace{1}_{\text{from } \delta \text{ length}} \underbrace{-2\varphi)T - (\varphi N)'}_{\text{from } \delta r, d\nu})'.$$

Theorem (with Fu, Kusner, Sullivan, Wrinkle 2006)

An explicit construction of a critical configuration of the Borromean rings with ropelength a definite integral which evaluates to ~ 58.0060.





Classification of critical curves without struts

In a kink-only critical curve, we have $\mathbb{S} = 0$, so

$$(1 - 2\varphi)T - (\varphi N)' \equiv V_0 = \text{constant}. \quad (1)$$

Notice that V_0 is some conserved vector along the curve. Differentiating, we show a vector is equal to 0. This yields

$$\varphi'' + (\kappa^2 - \tau^2)\varphi = \kappa^2 \quad (2)$$

$$\tau\varphi^2 = c \quad (3)$$

for some constant c. Since $\kappa = 1$, this is a system of ODE for τ and φ with initial conditions specified by c and $\varphi(0)$, and a constant solution $\varphi = \varphi_0(c)$.

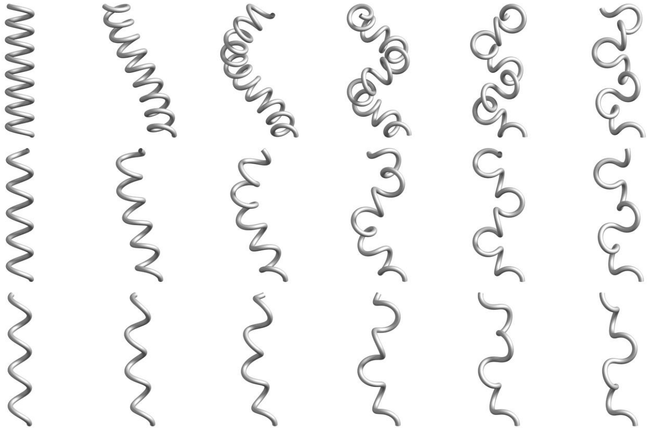
Pictures of solutions

$\varphi_0(c) \times$ 1   1.125   1.25   1.5   1.75   2

$c = \frac{1}{2}$

$c = 1$

$c = 3$



The general case.

We may assume $c \neq 0$, so φ is not always zero. Where $\varphi > 0$, we have $\tau = c/\varphi^2$, so (2) and (3) become the semilinear ODE

$$\varphi'' = \kappa^2(1 - \varphi) + \frac{c}{\varphi^3} := f_c(\varphi). \tag{4}$$

Lemma

All solutions of (4) are positive periodic functions.

Proof.

(4) is an autonomous system with integrating function

$$F(x, y) = \left(\frac{\kappa^2}{2}x^2 + \frac{1}{2}y^2\right) - \kappa^2 x + \frac{c^2}{2x^2} = \text{const}, \tag{5}$$

where $x = \varphi$ and $y = \varphi'$. □

The general case (continued).

Theorem (CFKSW (2008))

Any closed piecewise C^2 λ-critical curve with no strut force measure is a circle of radius $\lambda/2$.

Proof.

We have reduced to the case $\varphi > 0$ with period P. Note

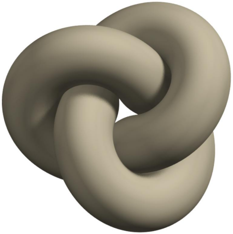
$$T \cdot V_0 = (1 - 2\varphi) - \varphi T \cdot N' = 1 - \varphi. \tag{6}$$

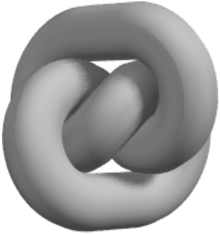
Solving (4) for $1 - \varphi$, we see $1 - \varphi = \frac{1}{\kappa^2}\varphi'' - \frac{c}{\kappa^2 \varphi^3}$. So we have

$$\int_0^P T \cdot V_0 \, \mathrm{d}s = \int_0^P \frac{1}{\kappa^2}\varphi'' - \frac{c}{\kappa^2 \varphi^3} \, \mathrm{d}s = -\frac{c}{\kappa^2} \int_0^P \varphi^{-3} \, \mathrm{d}s. \tag{7}$$

This $\neq 0$, since $c \neq 0$ and $\varphi > 0$. So over each period the curve moves a constant distance in the V_0 direction. □

Maybe how to find alternate critical configurations





Remark

This strategy can't be extended to find Gordian unknots, because the round circle already has a symmetry of every period.

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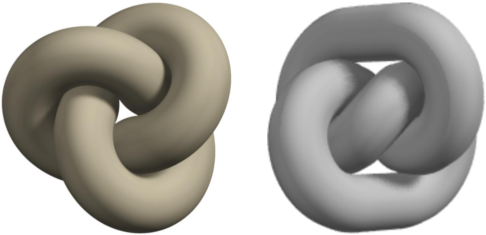

Theorem (with Fu, Kusner, Sullivan, Wrinkle, in preparation)

There is another critical configuration of 3_1 *with* 2*-fold symmetry.*

Proof.

The proof is based on a symmetric version of the criticality theorem. There should be a critical configuration with 3-fold and with 2-fold symmetry (there is no configuration with both symmetries.) □

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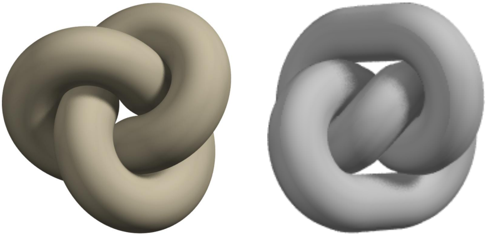

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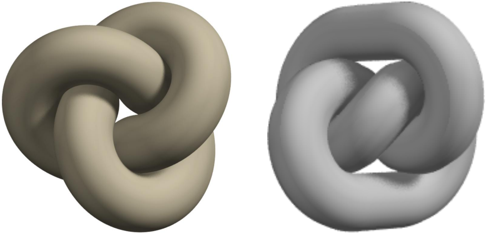
But not Gordian unknots . . .



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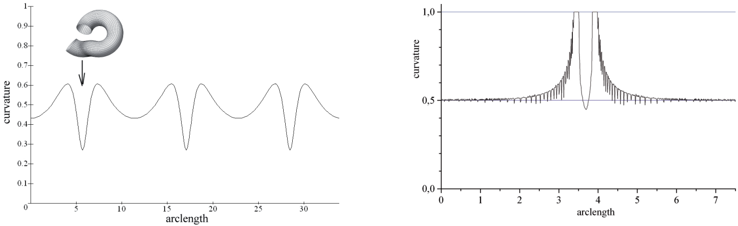
This strategy can't be extended to find Gordian unknots, because the round circle already has a symmetry of every period.

Open Question

Can you find an exact description of the shape of a tight knot?

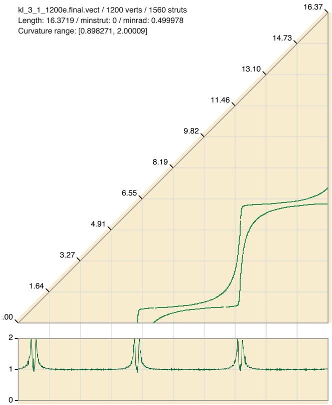
You have the balance theorem to work with and a tremendous amount of numerical data to help solve the structure. For instance, here's a plot of the curvature of the knot:

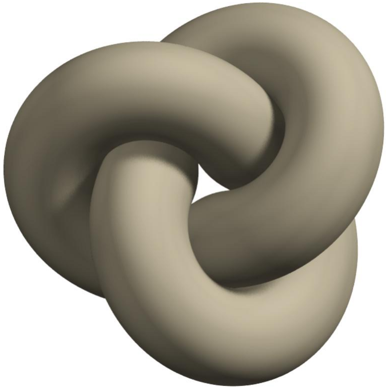


(Baranska, Pieranski, and Przybyl 2008)

Numerical Data on the Trefoil Knot

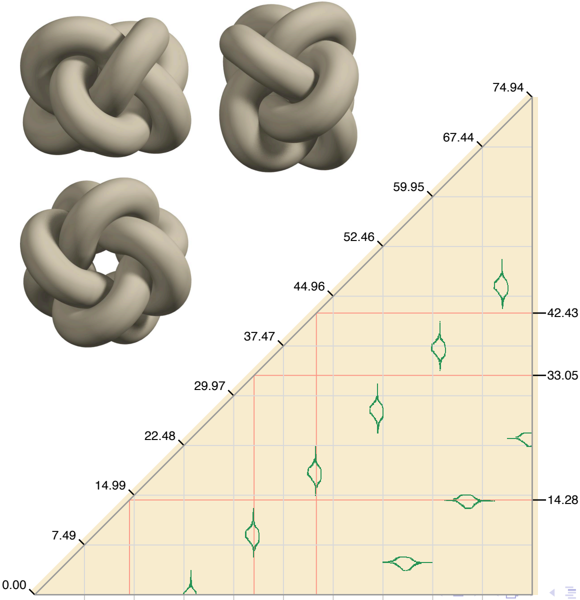
The struts are described by points on the (s, t) plane:





(with Ashton, Piatek, Rawdon 2005)

But 8_{18} might actually be easier to solve...



Conclusion: One last open problem...

Definition

The *writhe* of a space curve K is given by

$$\mathrm{Wr}(K) = \frac{1}{4\pi} \iint \frac{K'(s) \times K'(t) \cdot (K(s) - K(t))}{|K(s) - K(t)|^3} \, \mathrm{d}s \, \mathrm{d}t.$$

Open Question

For an unknot, is there a constant c so $\mathrm{Wr}(K) \leq c \, \mathrm{Rop}(K)$? This is not true for nontrivial knots, since $(n, n-1)$ torus have $\mathrm{Wr} \sim \mathrm{Rop}^{4/3}$.

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Thank you for inviting me!

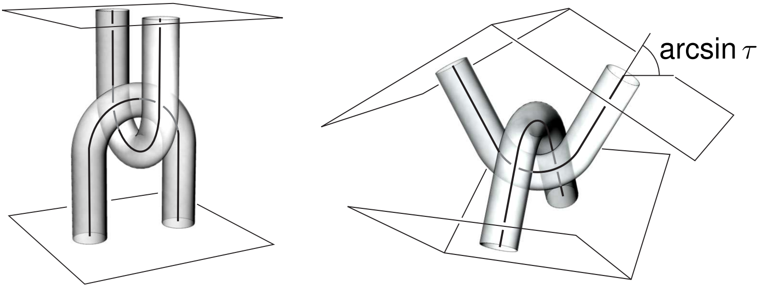
Thank you for inviting me! (And more movies if there's time . . .)
Slides on the web at:

`http://www.jasoncantarella.com/`

under "Courses" and "Geometric Knot Theory".

Another solution: Clasps

What happens when a rope is pulled over another?



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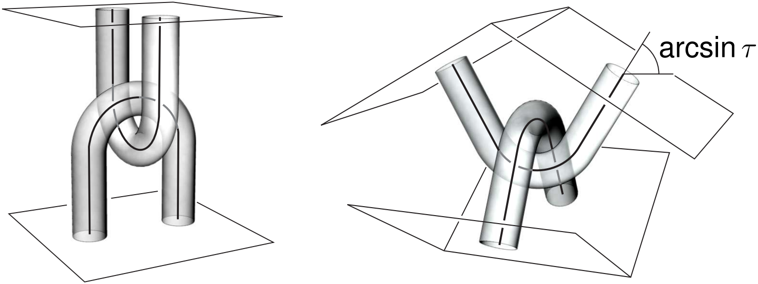
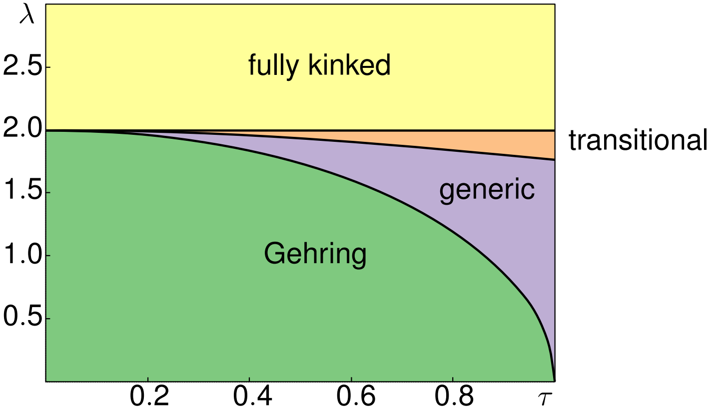
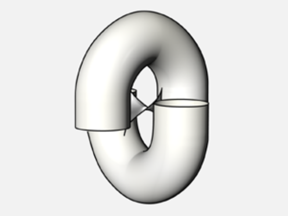


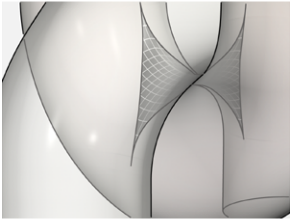
It depends on the angle (τ) and the stiffness (λ) of the rope.

Four types of clasps



Gehring clasp (CFSKW 2006)





- δ length balanced against strut force only.
- Curvature given explicitly, position as an elliptic integral.
- Small gap between the two tubes.
- Curvature unbounded at tip.

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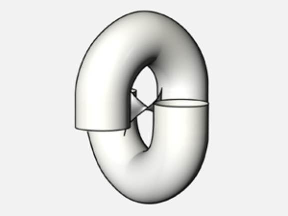

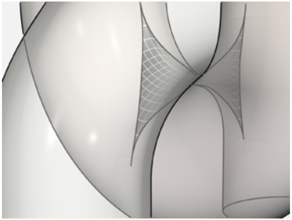

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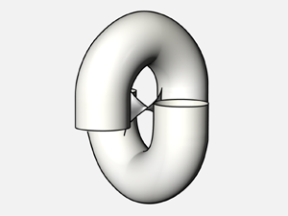

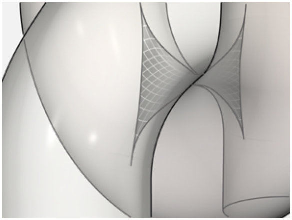

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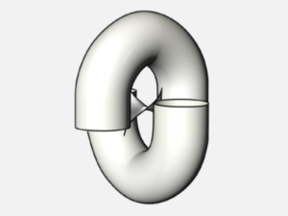

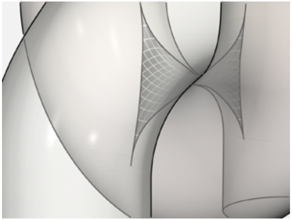

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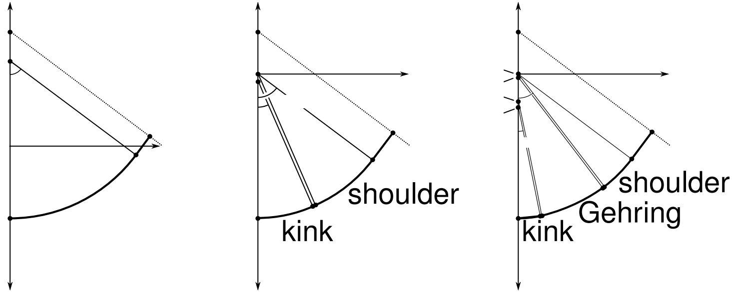
Kinked, Transitional, Generic Clasps



Kinked Clasp Transitional Clasp Generic Clasp