Introduction to Geometric Knot Theory 1: Knot invariants defined by minimizing curve invariants

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Main question

Every field has a main question:

Open Question (Main Question of Topological Knot Theory)

What are the isotopy classes of knots and links?

Open Question (Main Question of Global Differential Geometry)

What is the relationship between the topology and (sectional, scalar, Ricci) curvature of a manifold?

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Two Main Strands of Inquiry

Strand 1 (Knot Invariants defined by minima)

Given a geometric invariant of curves, define a topological invariant of knots by minimizing over all curves in a knot type. How are these invariants related?

Examples: Crossing number, bridge number, total curvature, distortion, braid index, Möbius energy, ropelength.

Strand 2 (Restricted Knot Theories)

Restrict attention to curves obeying additional geometric hypotheses. Is every knot type realizable in this class? What are the isotopy types among curves in this class?

Examples: Regular (nonvanishing curvature) isotopy, Legendrian and transverse knots, braid theory, polygonal knots, plumber's knots.

An example: crossing number

Definition

The crossing number of a knot or link is the minimum number of crossings in any projection of any configuration of the knot.

. . . that's why I hate crossing number. (attributed to J.H. Conway)

1. The unique knot of minimum crossing number is the unknot.
2. For any N, there only finitely many knot types with crossing number $< N$.

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Desirable Properties of Geometric Knot Invariants

Definition

A *geometric knot invariant* $MG([K])$ is a knot invariant defined by minimizing a geometric invariant $G(K)$ of curves over all curves K in a knot type $[K]$.

Definition

A geometric knot invariant $MG([K])$ is

- *basic* if the minimum of $MG([K])$ over all knot types is achieved uniquely for the unknot.
- *strong* if for each N there are only finitely many knot types with $MG([K]) < N$.

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An old and hard open question: distortion

We now give another geometric curve invariant.

Definition

The *distortion* of a curve γ is given by

$$\mathrm{Dist}(\gamma) = \max_{p,q \in \gamma} \frac{\text{distance from } p \text{ to } q \text{ on } \gamma}{\text{distance from } p \text{ to } q \text{ in space}}.$$

The corresponding geometric knot invariant is denoted MDist.

Open Question (Gromov, 1983)

Is there is a universal upper bound on MDist *for all knots?*

A popular question in the 80's, but pretty hard.

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What is known about distortion?

Theorem (Kusner and Sullivan 1999, Denne and Sullivan 2009)

MDist *is basic. The minimum distortion of an unknot is* $\pi/2$ *(the round circle) while the distortion of a nontrivial tame knot is at least* $5\pi/3$*.*

Theorem (Gromov 1978, O'Hara 1992)

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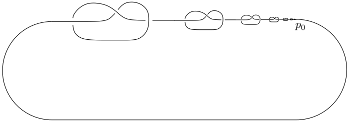

Conjecture: No upper bound on distortion for knots

To prove it:

1. Find a "sufficiently weak" knot invariant I that approaches infinity on $(n, n-1)$ torus knots. (Bridge number, crossing number, genus, etc are all ruled out already.)
2. Bound distortion below in terms of I.

Definition

The *hull number* of a knot type $[K]$ is the maximum N such that any curve K in $[K]$ has some point p so that any plane through p cuts K at least $2N$ times.

Theorem (Izmestiev 2006)

The hull number of a (p, q) *torus knot is at least* $(1/4)\min(p, q)$.

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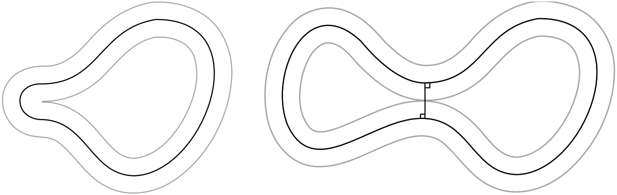
A geometric invariant of curves: Reach

Definition (Federer 1959)

The *reach* of a space curve is the largest ϵ so that any point in an ϵ-neighborhood of the curve has a unique nearest neighbor on the curve.



Idea

reach(K) *(also called thickness) is controlled by curvature maxima (kinks) and self-distance minima (struts).*

Ropelength

Definition

The ropelength of K is given by $\mathrm{Rop}(K) = \mathrm{Len}(K)/\,\mathrm{reach}(K)$.

Theorem (with Kusner, Sullivan 2002, Gonzalez, De la Llave 2003, Gonzalez, Maddocks, Schuricht, Von der Mosel 2002)

Ropelength minimizers (called tight knots) exist in each knot and link type and are $C^{1,1}$.

Open Question

What is the smoothness of a tight knot? Current examples suggest that such a knot is piecewise smooth but not C^2.

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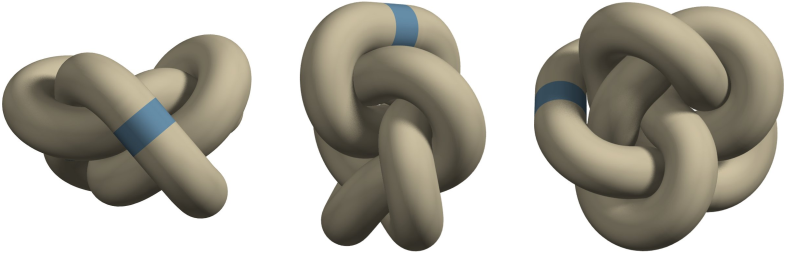
Examples: Why only piecewise smooth?



Theorem (with Fu, Kusner, Sullivan, Wrinkle 2009, cf. Gonzalez, Maddocks 2000, Schuricht, Von der Mosel 2003)

Any open interval of a tight knot either: contains an endpoint of a strut, has curvature 1 *almost everywhere, or is a straight line segment.*

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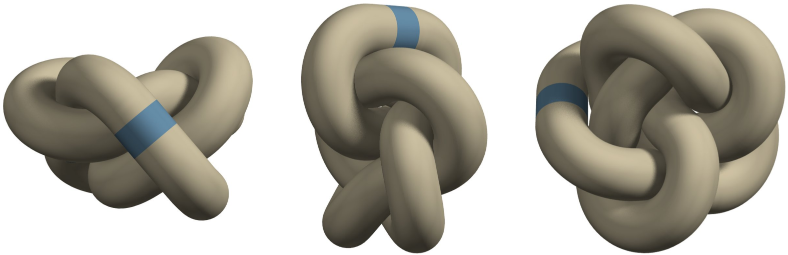

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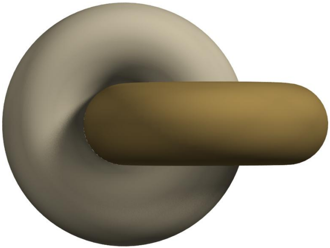
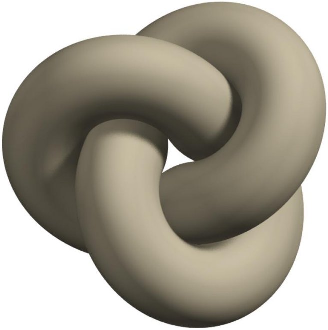
The Tight Hopf Link and Trefoil Knot

The Hopf Link 2_1^2
$\text{Rop}([2_1^2]) = 8\pi$
with Kusner, Sullivan 2002



The Trefoil Knot 3_1
$31.32 \leq \text{Rop}([3_1]) \leq 32.743175$
Denne, Diao and Sullivan 2006
Baranska, Przybyl, Pieranski 2008



Lower Bounds on Ropelength

Theorem (Diao 2006)

$$\mathrm{Rop}(K) \geq \frac{1}{2}\left(17.334 + \sqrt{17.334^2 + 64\pi\ \mathrm{Cr}(K)}\right).$$

Corollary

Ropelength is basic and strong.

Proof.

The ropelength of a tight unknot is $4\pi = 12.566$, less than any knot of higher crossing number. All knots with Rop $< N$ have

$$\mathrm{Cr} < \frac{0.000125}{\pi} N(500N - 8667).$$

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More consequences of this Cr / Rop bound.

Corollary

Hopf link ($\mathrm{Rop} = 25.1327$*) is the tightest nontrivial link.*

Proof.

Evaluating the formula in a few cases,

$\mathrm{Cr}(K)$	3	4	5	...	10	11
$\mathrm{Rop}(K) \geq$	23.698	25.286	26.735	...	32.704	33.73

So only $\mathrm{Rop}(3_1)$ could be lower than $\mathrm{Rop}(2_1)$. But DDS show $\mathrm{Rop}(3_1) \geq 31.32 > 25.137$. □

Open Question

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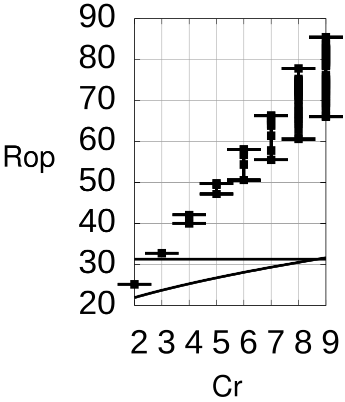
Ropelength and Crossing Number vs Data

Open Question

Find effective Rop *bounds for simple* (< 10 *crossing) knots.*



Cr	Rop	Links
3	32.74	3_1
4	$[40.01, 42.09]$	4^2_1, 4_1
5	$[47.20, 49.77]$	5_1, 5^2_1
6	$[50.57, 58.1]$	6^3_3, 6^3_2
7	$[55.53, 66.33]$	7^2_7, 7^2_6
8	$[60.58, 77.83]$	8^3_7, 8^3_4
9	$[66.06, 85.47]$	9^2_{49}, 9^4_1

Ropelength and Crossing Number

Theorem (Buck, Simon 1999, Diao, Ernst, Yu 2003)

There exist constants so $c_1 \operatorname{Cr}^{3/4}(K) \leq \operatorname{Rop}(K) \leq c_2 \operatorname{Cr}^{3/2}(K)$.

Proof (sketch) of 3/4 power lower bound.

Scale the knot so $\operatorname{reach}(K) = 1$. Then $\operatorname{Rop}(K) = \operatorname{Len}(K)$.

$$\operatorname{Cr}(K) \leq \operatorname{ACr}(K) = \frac{1}{4\pi} \iint \frac{|K'(s) \times K'(t) \cdot (K(s) - K(t))|}{|K(s) - K(t)|^3} \, \mathrm{d}s \, \mathrm{d}t$$
$$\leq \frac{1}{4\pi} \iint \frac{1}{|K(s) - K(t)|^2} \, \mathrm{d}s \, \mathrm{d}t.$$

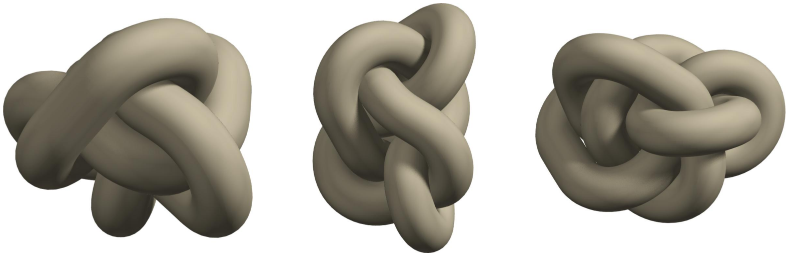
Now we estimate this integral above in terms of $\operatorname{Len}(K)$. □

Proof (sketch) of $3/4$ power lower bound

We start by estimating

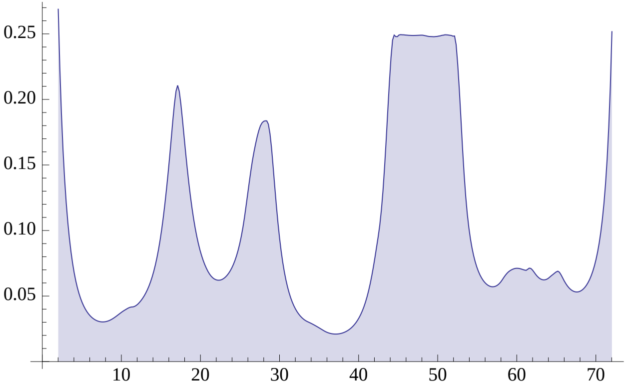
$$\int_{d(s,t)>2} \frac{1}{|K(s) - K(t)|^2} \,\mathrm{d}s$$

where $d(s,t)$ is the arclength distance along K. Our example plots will come from this 9_{49} knot:



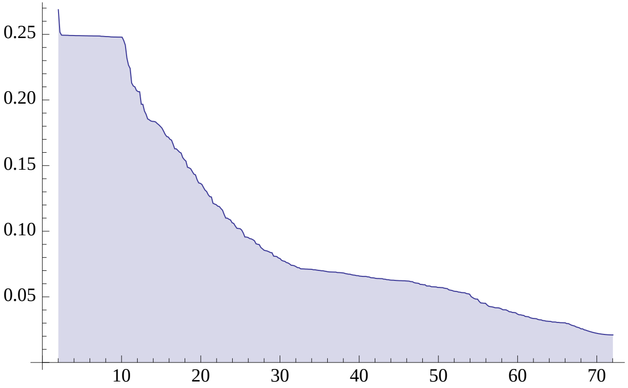
Proof (sketch) of $3/4$ power lower bound

Here is a graph of the inverse square distance from $K(0)$ to $K(s)$ for the 9_{49} knot above:



Proof (sketch) of 3/4 power lower bound

Without changing the integral, we can take a monotone rearrangement of the function:



Proof (sketch) of $3/4$ power lower bound

A (possibly disconnected) section of tube of total arclength s has volume πs. If that section of tube is within distance r of the origin, then this tube is all packed in the sphere of radius $r + 1$, which has volume $(4/3)\pi(r+1)^3$. Assuming $r > 2$,

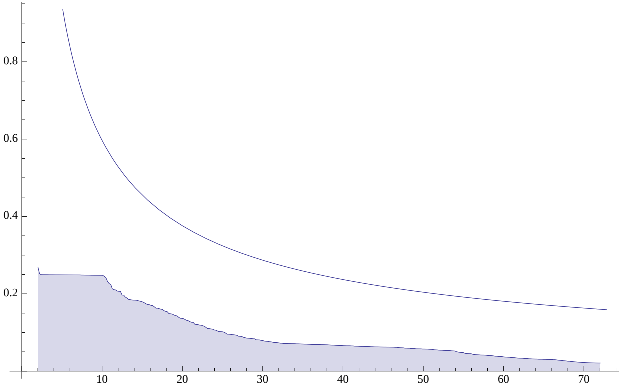
$$\pi s < \frac{4}{3}\pi\left(r+1\right)^3 < 4.5r^3,$$

we can rearrange to get

$$2.73\, s^{-\frac{2}{3}} > \frac{1}{r^2}.$$

Proof (sketch) of $3/4$ power lower bound

This estimate shows that our rearranged distance function is less than $2.73\ s^{-2/3}$ (when $1/r^2 < 0.25$):



Proof (sketch) of 3/4 power lower bound

Integrating over $[0, \mathrm{Len}(K)]$, we get

$$\int_{d(s,t)>2} \frac{1}{|K(s) - K(t)|^2} \, \mathrm{d}s < \int_0^{\mathrm{Rop}(K)} 2.77 \, s^{-2/3} \, \mathrm{d}s$$
$$< 8.177 \, \mathrm{Rop}(K)^{1/3}.$$

and so

$$\mathrm{Cr}(K) \leq \frac{1}{4\pi} \iint \frac{1}{|K(s) - K(t)|^2} \, \mathrm{d}s \, \mathrm{d}t < 0.651 \, \mathrm{Rop}(K)^{4/3}.$$

and taking care of the pairs $d(s,t) < 2$ with another argument,

$$1.38 \, \mathrm{Cr}^{3/4} + (\text{lower order terms}) \leq \mathrm{Rop}(K).$$

Open Question: What's the best bound of this type?

The actual bound of Buck and Simon is

Theorem (Buck and Simon 1999)

$$\mathrm{Rop}(K) \geq 2.205\,\mathrm{Cr}(K)^{3/4}$$

One can easily improve the argument above to

$$\mathrm{Rop}(K) \geq 2.5357\ \mathrm{Cr}(K)^{3/4} + \text{(lower order terms)}$$

but the lower order terms are significant.

Open Question

What is the largest c_1 so that $c_1\,\mathrm{Cr}(K)^{3/4} \leq \mathrm{Rop}(K)$ for all K?

A remark

Theorem (Diao 2006)

$$\operatorname{Rop}(K) \geq \frac{1}{2}\left(17.334 + \sqrt{17.334^2 + 64\pi \operatorname{Cr}(K)}\right).$$

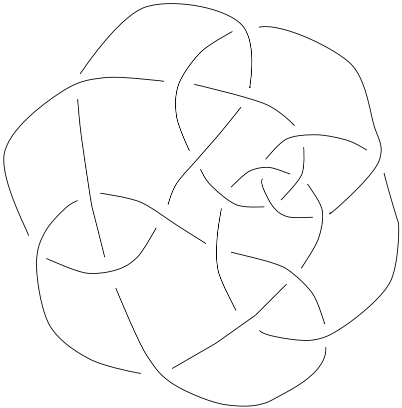
is also proved by bounding

$$\begin{aligned}\operatorname{Cr}(K) \leq \operatorname{ACr}(K) &= \frac{1}{4\pi}\iint \frac{|K'(s) \times K'(t) \cdot (K(s) - K(t))|}{|K(s) - K(t)|^3}\,\mathrm{d}s\,\mathrm{d}t \\ &\leq \frac{1}{4\pi}\iint \frac{1}{|K(s) - K(t)|^2}\,\mathrm{d}s\,\mathrm{d}t.\end{aligned}$$

but in Diao's proof the lower order terms dominate the bound.

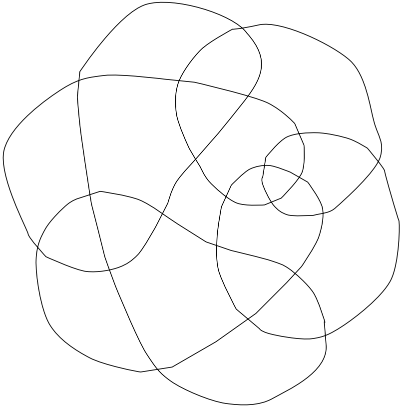
Proof (sketch) of $3/2$ power upper bound

We must find an algorithm for constructing "short" embeddings of knots, such as this one: $21_{4,385,281}$



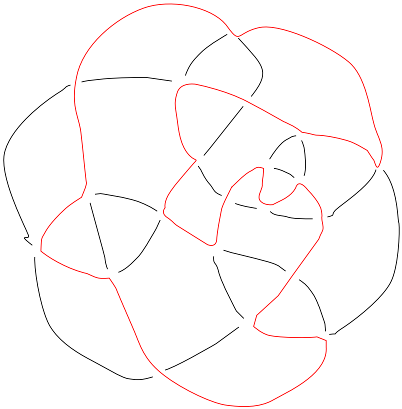
Proof (sketch) of 3/2 power upper bound

Start by converting it to a planar 4-regular graph:



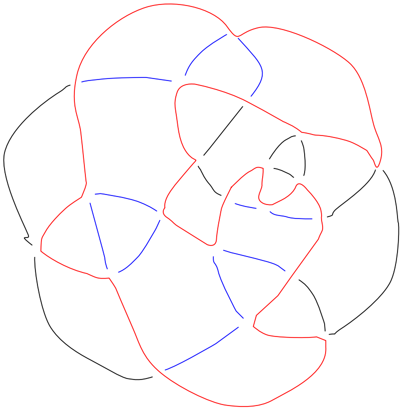
Proof (sketch) of $3/2$ power upper bound

We can arrange for such graphs to always be Hamiltonian (by adding extra verts if needed):



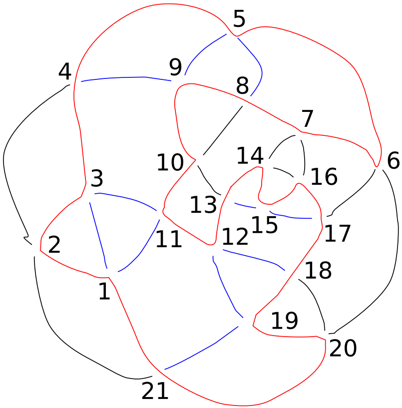
Proof (sketch) of 3/2 power upper bound

The edges not on the Hamiltonian circuit are either “inner” or “outer” edges.



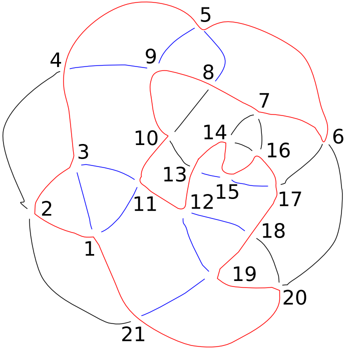
Proof (sketch) of 3/2 power upper bound

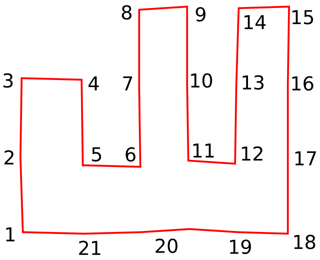
Now we can start the embedding. First, number the vertices to help us keep track:



Proof (sketch) of $3/2$ power upper bound

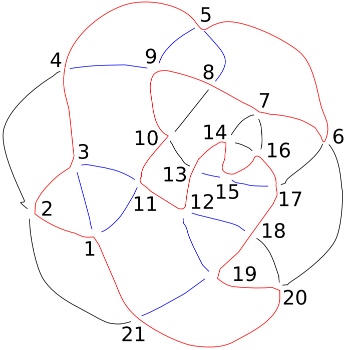
We can lay out the Hamiltonian circuit in a compact manner on a $\sqrt{\mathrm{Cr}} \times \sqrt{\mathrm{Cr}}$ grid in the *xy* plane with length $\sim$ Cr.

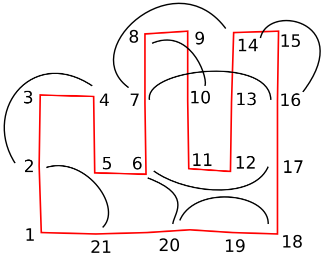




Proof (sketch) of $3/2$ power upper bound

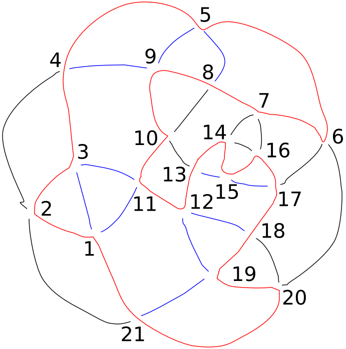
The “outer edges” are embedded above the *xy* plane. They have length $\sim \sqrt{\text{Cr}}$, and there are at most Cr of them.

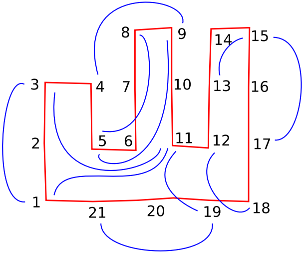




Proof (sketch) of $3/2$ power upper bound

The “inner edges” are embedded below the *xy* plane. They also have length $\sim \sqrt{\mathrm{Cr}}$, and there are at most Cr of them.



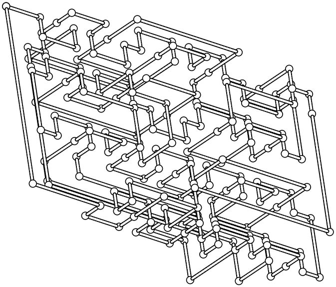


Proof (sketch) of 3/2 power upper bound

This gives a total ropelength $\sim \mathrm{Cr}^{3/2}$ (modulo plenty of details). The actual upper bound is

$$\mathrm{Rop}(K) \leq 272\ \mathrm{Cr}(K)^{3/2} + 168\ \mathrm{Cr}(K) + 44\sqrt{\mathrm{Cr}(K)} + 22.$$

Diao et. al. have implemented this algorithm, producing:



The asymptotic relationship between Rop and Cr

Theorem (with Kusner, Sullivan 1998, Diao, Ernst 1998)

There are examples of infinite families of knots with $\mathrm{Rop}(K_n) \sim \mathrm{Cr}^p(K_n)$ *for every p between* $3/4$ *and* 1.

Example for $p = 3/4$.

In a solid torus of radii r and R "cabled" with unit tubes forming an $(n, n-1)$ torus knot, $r \sim \sqrt{n}$ and $R \sim r$.

$$\mathrm{Rop} \sim nR \sim n^{3/2}, \quad \mathrm{Cr} = n(n-2) \sim n^2, \quad \mathrm{Rop} \sim \mathrm{Cr}^{3/4}.$$

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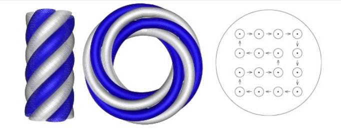

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An example family of links with $\mathrm{Rop}(L) = a\,\mathrm{Cr}(L) + b$

Theorem (with Kusner, Sullivan 2002)

The minimum ropelength of a chain L_n of n links is

$$\mathrm{Rop}(L_n) = (4\pi + 4)n - 8.$$

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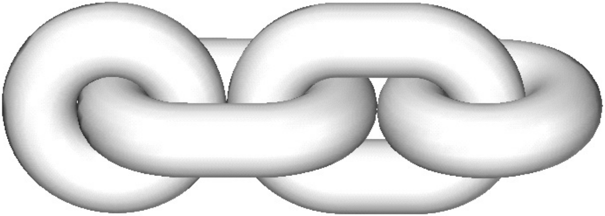

The inner “stadium curves” have length $4\pi + 4$ while the end rings have length 4π. Proof discussed later.

Open Questions: Ropelength and Crossing Number

1. Is there any family of links with $\text{Rop}(L_n) \sim \text{Cr}(L_n)^p$ for $p > 1$?
2. Can you find an upper bound so that $\text{Rop}(K) \leq c_2 \, \text{Cr}(K)^p$ for $p < 3/2$? The graph embedding literature suggests that a better bound should be possible.
3. The example family with $\text{Rop}(L_n) \sim \text{Cr}(L_n)^{3/4}$ was very nonalternating. Is it true that $\text{Rop}(L_n) \geq c \, \text{Cr}(L_n)$ for alternating knots?
4. The simple chain of 3 links has ropelength $12\pi + 4$. It is a connect sum of two Hopf links, each with ropelength 8π. Is is always true that

$$\text{Rop}(K_1 \# K_2) \leq \text{Rop}(K_1) + \text{Rop}(K_2) - (4\pi - 4)?$$

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Thank you for coming!

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`http://www.jasoncantarella.com/`

under “Courses” and “Geometric Knot Theory”.

Topics for Lecture 2:

1. Ropelength bounds in terms of other knot invariants.
2. Computation of approximate ropelength minimizers.
3. Gordian unknots and local minima for ropelength.
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