Statistics on G/S manifolds.

We now turn to the question of defining statistics on our manifolds. Suppose we measure n points on $G_k(\mathbb{C}^n)$. What is their mean? How can we decide if they fit to an error distribution around the mean? What kind of errors might we expect? What is the variance of the observations? We are going to start with the circle as $V_1(\mathbb{R}^2)$.

Definition. Given unit vectors $\vec{X}_1, \dots, \vec{X}_n$ with angles $\theta_1, \dots, \theta_n$ on S^1, we say their <u>mean direction</u> $\bar{\theta}$ is the direction of the center of mass $\frac{1}{n}(\vec{X}_1 + \dots + \vec{X}_n) = \overline{X}$.

Since

$$\overline{X} = \frac{1}{n} \sum (\cos\theta_i, \sin\theta_i)$$

we can define

$$\overline{C} = \frac{1}{n} \sum \cos\theta_i , \quad \overline{S} = \frac{1}{n} \sum \sin\theta_i$$

and let

$$\overline{C} = \overline{R} \cos\overline{\theta}, \quad \overline{S} = \overline{R} \sin\overline{\theta}$$

where

$$\overline{R} = \sqrt{\overline{C}^2 + \overline{S}^2}.$$

We can use $\bar{R}$ as a measure of the dispersion of our data:

$$V = 1 - \bar{R}$$

is the <u>circular variance</u>, which is zero when all data points are the same, 1 when they are "evenly" distributed around the circle.

If we measure dispersion about α by

$$D(\alpha) = \frac{1}{n} \sum 1 - \cos(\theta_i - \alpha).$$

Lemma. $D(\alpha)$ is minimized when $\alpha = \bar{\theta}$, and $D(\bar{\theta}) = V$.

This is just like the definition of (sample) variance for linear data as the minimum value of

$$D(u) = \frac{1}{n}\sum (x_i - u)^2,$$

which occurs at $u = \bar{x}$. the mean

We can define <u>higher moments</u> of circular data by

Definition. The p-th <u>trignometric moment</u> about the mean direction $\bar{\theta}$ is given by

$$m_p = a_p + i b_p, \quad a_p = \frac{1}{n}\sum \cos p(\theta_i - \bar{\theta})$$
$$b_p = \frac{1}{n}\sum \sin p(\theta_i - \bar{\theta}).$$

These should look suspicious if you know a bit of harmonic analysis and they are.

(cumulative)

We can define the distribution function of a circular distribution by

$$F(x) = P(0 < \Theta \leq x), \quad (0 \leq x \leq 2\pi)$$

and

$$F(x+2\pi) - F(x) = 1 \text{ for any } x.$$

If F is absolutely continuous, there is a corresponding pdf $f(x) = dF$ so that

$$\int_{\alpha}^{\beta} f(x)\,dx = F(\beta) - F(\alpha)$$

for any α, β. We define

$$\alpha_p = E(\cos p\Theta) = \int_0^{2\pi} \cos p\Theta \; dF$$

$$\beta_p = E(\sin p\Theta) = \int_0^{2\pi} \sin p\Theta \; dF$$

$$\varphi_p = \alpha_p + i\beta_p$$

and call the sequence φ_p of complex numbers the <u>characteristic function</u> of F. Of course, these are the Fourier coefficients of dF:

$$dF(\theta) = \frac{1}{2\pi} \sum_{p=-\infty}^{\infty} \varphi_p e^{-ip\theta}$$

The series (at right) converges if $\sum (\alpha_p^2 + \beta_p^2)$ does.

~~Example~~.

An easy use of the characteristic function is the observation that the c.f. of a sum is the product of the c.f.s of the summands:

$\theta + \theta'$ has c.f. $\varphi_p \varphi_p'$ if

θ has c.f. φ_p and θ' has c.f. φ_p'

We can use this to prove

Proposition. If Θ is a uniform r.v. and Θ' is any r.v. on S^1 then $\Theta+\Theta'$ is uniform on S^1.

Proof. Θ has c.f. $\varphi_p = 0$ for $p \neq 0$, $\varphi_0 = 1$, so $\Theta + \Theta'$ has cf. $\varphi_p \varphi_p' = 0$ for $p \neq 0$, $\varphi_0 \varphi_0' = 1$, as desired. $\square$.

We now introduce the standard family of circular distributions on S^1; the von Mises K, μ distributions.

We recall that the Von Mises distribution is given by

$$g(\theta; \mu, K) = \frac{1}{2\pi I_0(K)} e^{K\cos(\theta - \mu)}$$

where

$$I_0(K) = \frac{1}{2\pi} \int_0^{2\pi} e^{K\cos(\theta-\mu)} \, d\theta.$$

The characteristic function is given by

$$\beta_p = E[\sin p(\theta - \mu)] = 0,$$

since $g(\theta-\mu) = g(\mu-\theta)$, and

$$\begin{aligned} \alpha_p &= E[\cos p(\theta-\mu)] \\ &= \frac{I_p(K)}{I_0(K)}, \quad \text{where } I_p(K) = \frac{1}{2\pi}\int_0^{2\pi} \cos p\theta \, e^{K\cos\theta} \, d\theta. \end{aligned}$$

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If we compute the mean resultant length for the von Mises μ, K, we ~~give~~ get

$$\bar{R} = I_1(K)/I_0(K).$$

~~Theorem.~~

Definition. The entropy of a distribution with pdf $f(\theta)$ is given by

$$H(f) = -\int_0^{2\pi} f(\theta)\log f(\theta)\,d\theta$$

Theorem (Mardia, 1972) The maximum entropy distribution on S^1 with mean μ and mean resultant $\bar{R}$ is the von Mises distribution with $I_1(K)/I_0(K) = \bar{R}$.