The Schubert Calculus.

We now apply these techniques in the case where there are infinitely many solutions.

Definition. The subset $\Omega(A_0, \ldots, A_{k-1})$ of $G_k(\mathbb{C}^n)$ is defined by polynomial equations and so is a sub<u>variety</u> of $G_k(\mathbb{C}^n)$; in particular it is an orientable submanifold.

We can generate a corresponding homology class by taking the fundamental class and including it into $H_*(G_k(\mathbb{C}^n); \mathbb{Z})$, then applying Poincaré duality to get a class in $H^*(G_k(\mathbb{C}^n); \mathbb{Z})$.

showing that they represent the same cohomology class. $\square$

We can now give two main theorems:

Theorem 1. Considered additively, $H^*(G_k(\mathbb{C}^n);\mathbb{Z})$ is a free abelian group, and the Schubert cycles determine a basis.

In fact, we know more. We first claim:

Proposition. The dimension of $\Omega(A_0,..,A_{k-1})$ is $\Sigma(a_i - i)$.

We can represent $\Omega(A_0,..,A_k)$ as a space of $k\times n$ matrices.

Fact: The cohomology class associated to $\Omega(A_0,\dots,A_{k-1})$ is determined by the integers $a_0=\dim(A_0),\dots,a_{k-1}=\dim(A_{k-1})$ and it is called the <u>Schubert cycle</u> associated with $a_0,\dots,a_{k-1}$.

Proof. We ~~said~~ claimed that any flag ~~determined by~~ with dimensions $a_0,\dots,a_{k-1}$ was ~~equivalent~~ mapped to the standard flag by a nonsingular linear transformation, and (of course) that every nonsingular linear transformation of $\mathbb{C}P^N$ takes the standard flag to another flag.

Thus $\Omega(A_0,\dots,A_{k-1})$ is homotopic to $\Omega(B_0,\dots,B_{k-1})$ if $\dim A_i=\dim B_i$, In this space,

$$L = \begin{bmatrix} p_0(0) & \cdots & p_0(a_0) & \cdots & 0 & & & 0 \\ p_i(0) & \cdots & \cdots & \cdots & p_i(a_i) & 0 & \cdots & 0 \\ p_k(0) & & & & & & p_k(a_k) & 0 \end{bmatrix}$$

Now for ~~each~~ an open set of such L, there is a unique linear transformation of $\mathbb{C}^k$ so that

$$L' = \begin{bmatrix} p_0'(0) & \cdots & 1 & 0 & & & & 0 \\ p_i'(0) & \cdots & 0 & \cdots & 1 & & & \\ & & \vdots & & 0 & & & \\ p_k'(0) & \cdots & 0 & \cdots & 0 & 1 & 0 \cdots & 0 \end{bmatrix}$$

We will take this as <u>the</u> representative of $[L] \in G_k(\mathbb{C}^n)$.

The space of such L' clearly has dimension $(a_0 - 0) + (a_1 - 1) + (a_2 - 2) + \ldots + (a_{k-1} - (k-1))$, as desired. Since this is an open subset of the manifold $\Omega(A_0, \ldots, A_{k-1})$ it must have the same dimension.

We can now say more:

Theorem. Each even dimensional integral cohomology group $H^{2p}(G_k(\mathbb{C}^n);\mathbb{Z})$ is a free abelian group and the Schubert cycles $\Omega(a_0,\ldots,a_{k-1})$ with

$$\sum a_i - i = k(n-k) - p$$

form a basis. Each odd dimensional cohomology group is zero.

Example. Consider $G_2(\mathbb{C}^4)$. There are nonzero groups:

$$H^8(G_2(\mathbb{C}^4);\mathbb{Z}) = \langle \Omega(0,1) \rangle$$

$$H^6(G_2(\mathbb{C}^4);\mathbb{Z}) = \langle \Omega(1,3) \rangle$$

$$H^4(G_2(\mathbb{C}^4);\mathbb{Z}) = \langle \Omega(0,3), \Omega(1,2) \rangle$$

$$H^2(G_2(\mathbb{C}^4);\mathbb{Z}) = \langle \Omega(1,3) \rangle$$

$$H^0(G_2(\mathbb{C}^4);\mathbb{Z}) = \langle \Omega(2,3) \rangle.$$

Further, we know:

$\cancel{\Omega(0,1)}$ Poincaré dual of:

$\Omega(0,1)$ point

$\Omega(2,3)$ entire space

$\Omega(1,3)$ hyperplane section of $G_2(\mathbb{C}^4) \subset \mathbb{CP}^6$.

Now given a flag

$$A_0 \subsetneq \cdots \subsetneq A_{K-1} \text{ with dims } a_0, \ldots, a_{K-1}$$

we can construct a dual flag

$$A_{K-1}^{\perp} \subsetneq A_{K-2}^{\perp} \subsetneq \cdots \subsetneq A_0^{\perp} \text{ with dims } n-a_{K-1}, \ldots, n-a_0$$

It's to be hoped that

$$\Omega(a_0, \ldots, a_{K-1}) \text{ and } \Omega(n-a_{K-1}, \ldots, n-a_0)$$

are related in a nice way!

In fact,

Proposition. $\Omega(a_0, \ldots, a_{k-1}) \cup \Omega(n-a_{k-1}, \ldots, n-a_0) = [G_k(\mathbb{C}^n)]$

so these are dual classes in the sense of Poincaré duality.