Schubert Calculus III

We have now proved:

Theorem. The Plücker relations determine the image of $G_k(\mathbb{C}^n)$ in $\mathbb{P}^N$ under the Plücker embedding.

We start with some examples.

Example. $G_2(\mathbb{R}^4)$.

There is ~~only~~ ~~only~~ one ~~pair of~~ sequences $j=(1)$ and $\ell=(234)$, which yields the Plücker relation

$$p(12)p(34)-p(13)p(24)+p(14)p(23)=0.$$

The sequences $j=(2)$, $\ell=(134)$ yield~~s~~

$$p(21)p(34)-p(23)p(14)+p(24)p(13)=0$$

which is just

$$-p(12)p(34)-p(23)p(14)+p(24)p(13)=0$$

which is actually <u>not</u> distinct (it's just a change of sign from the previous).

If we view $G_2(\mathbb{R}^4)$ as the space of perimeter 2 polygons in $\mathbb{R}^2$ with 4 edges, we have the interpetation:

$$L = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \end{bmatrix}$$

$$p(12) = \det\begin{bmatrix} v_1 & v_2 \end{bmatrix} = v_1 \times v_2 = |v_1||v_2| \sin\theta_{12}$$

$$p(ij) = \det[v_i \; v_j] = v_i \times v_j = |v_i||v_j| \sin\theta_{ij}$$

Now if we convert $v_i \to w_i$ via the Hopf map, we transform the angle θ_{ij} to $\varphi_{ij} = 2\theta_{ij}$, since the Hopf map is the complex squaring operation.

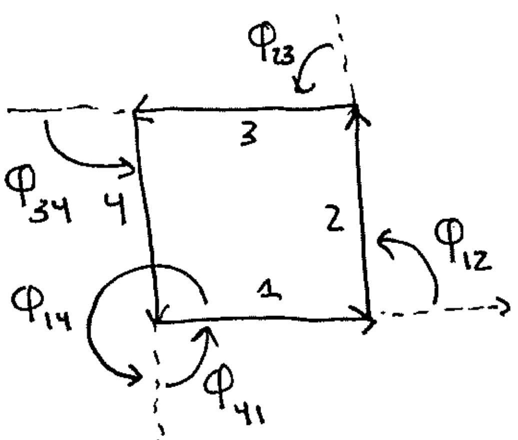
We can write the plücker relation as

$$|v_1||v_2|\sin\theta_{12}\;|v_3||v_4|\sin\theta_{34}$$
$$-\;|v_1||v_3|\sin\theta_{13}\;|v_2||v_4|\sin\theta_{24}$$
$$|v_1||v_4|\sin\theta_{14}\;|v_2||v_3|\sin\theta_{23} = 0$$

We see that we can cancel $|v_1||v_2||v_3||w_4|$ and get a relation

$$\sin\frac{\varphi_{12}}{2}\sin\frac{\varphi_{34}}{2} - \sin\frac{\varphi_{13}}{2}\sin\frac{\varphi_{24}}{2} + \sin\frac{\varphi_{14}}{2}\sin\frac{\varphi_{23}}{2} = 0.$$

Some easy checks are:



$\varphi_{13} = \pi$
$\varphi_{24} = \pi$

so we get

$$\sin\frac{\pi}{4}\sin\frac{\pi}{4} - \sin\frac{\pi}{2}\sin\frac{\pi}{2} + \sin\frac{3\pi}{4}\sin\frac{\pi}{4} = 0$$

which is certainly true.

Harrison Chapman points out that we now rewrite these terms using

$$\sin A \sin B = \tfrac{1}{2}\left(\cos(A-B) - \cos(A+B) \right)$$

to get

$$\frac{1}{2}\Bigg[\cos\left(\frac{\varphi_{12} - \varphi_{34}}{2} \right) - \cos\left(\frac{\varphi_{12} + \varphi_{34}}{2} \right)$$

$$- \cos\left(\frac{\varphi_{13} - \varphi_{24}}{2} \right) + \cos\left(\frac{\varphi_{13} + \varphi_{24}}{2} \right)$$

$$\cos\left(\frac{\varphi_{14} - \varphi_{23}}{2} \right) - \cos\left(\frac{\varphi_{14} + \varphi_{23}}{2} \right) \Bigg] = 0.$$

We now claim that since the φ_{ij} are angles between vectors, we must have $\varphi_{ij} = -\varphi_{ji}$ and

$$\varphi_{ij} + \varphi_{jk} \pm \varphi_{ik}$$

with these relations, we have

$$\varphi_{12} - \varphi_{34} = (\varphi_{14} + \varphi_{42}) - \varphi_{34} = (\varphi_{14} + \varphi_{43}) + \varphi_{42}$$
$$= \varphi_{13} - \varphi_{24},$$

so

$$\cos\left(\frac{\varphi_{12} - \varphi_{34}}{2}\right) - \cos\left(\frac{\varphi_{13} - \varphi_{24}}{2}\right) = 0.$$

We then have

$$\varphi_{12} + \varphi_{34} = (\varphi_{13} + \varphi_{32}) + \varphi_{34} = (\varphi_{13} + \varphi_{34}) + \varphi_{32}$$
$$= \varphi_{14} - \varphi_{23}$$

so

$$-\cos\left(\frac{\varphi_{12} + \varphi_{34}}{2}\right) + \cos\left(\frac{\varphi_{14} - \varphi_{23}}{2}\right) = 0$$

Lastly, we have

$$\varphi_{13} + \varphi_{24} = (\varphi_{12} + \varphi_{23}) + \varphi_{24} = (\varphi_{12} + \varphi_{24}) + \varphi_{23}$$
$$= \varphi_{14} + \varphi_{23}$$

so

$$\cos\left(\frac{\varphi_{13} + \varphi_{24}}{2}\right) - \cos\left(\frac{\varphi_{14} + \varphi_{23}}{2}\right) = 0.$$

The next question is how to extend this to space curves. We start by using the fact that the map from quaternions $\leftrightarrow$ $Mat_{2\times 2}(\mathbb{C})$ given by

$$\eta(q) = \eta(a+bj) = \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix}$$

is a homomorphism. ~~This means that~~
If we take the matrix

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = M(q_1, q_2)$$

then we have

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = a_1 b_2 - a_2 b_1$$

Now if we take $q_1 \bar{q}_2$ as a quaternion product, we know that $\bar{q}_2$ is given by

$\bar{q}_2 = \bar{a}_2 - b_2 j$, so

$$q_1\bar{q}_2 = \begin{bmatrix} a_1 & b_1 \\ -\bar{b}_1 & \bar{a}_1 \end{bmatrix}\begin{bmatrix} \bar{a}_2 & -b_2 \\ \bar{b}_2 & a_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1\bar{a}_2 + b_1\bar{b}_2 & -a_1 b_2 + b_1 a_2 \\ -\bar{b}_1\bar{a}_2 + \bar{a}_1\bar{b}_2 & \bar{b}_1 b_2 + \bar{a}_1 a_2 \end{bmatrix}$$

$$= \langle (a_1, b_1), (a_2, b_2) \rangle - (a_1, b_1) \times (a_2, b_2)\, j$$

$$= \langle v_1, v_2 \rangle - \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} j$$

Now recall that as rotation matrices unit quaternions are given by

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\left(n_1 i + n_2 j + n_3 K\right)$$

where θ is the angle of rotation and n_1, n_2, n_3 the axis of rotation. Thus as <u>matrices</u>

$$\bar{q} = \cos\frac{\theta}{2} - \sin\frac{\theta}{2}\left(n_1 i + n_2 j + n_3 K\right)$$

$$= \cos\left(-\frac{\theta}{2}\right) + \sin\left(-\frac{\theta}{2}\right)(n_1 i + n_2 j + n_3 k),$$

which is to say that $\bar{q}$ is the inverse element of SO(3). Now

$$q_2\bar{q}_1 = \begin{bmatrix} a_2 & b_2 \\ -\bar{b}_2 & \bar{a}_2 \end{bmatrix}\begin{bmatrix} \bar{a}_1 & -b_1 \\ \bar{b}_1 & a_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_2\bar{a}_1 + b_2\bar{b}_1 & -a_2 b_1 + b_2 a_1 \\ -\bar{a}_1\bar{b}_2 + \bar{a}_2\bar{b}_1 & -\bar{b}_2 b_1 + \bar{a}_2 a_1 \end{bmatrix}$$

$$= \langle (a_2, b_2), (a_1, b_1) \rangle + \det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} j$$

We can now observe that for <u>non</u>-unit quaternions, we have

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = |q_1||q_2| \sin\frac{\theta_{12}}{2}(n_2 + n_3 i)$$

where θ_{12} is the angle of rotation

But wait! There's a fly in the ointment! We have

$$\begin{tikzcd} q_1 \arrow[d, "\text{Hopf}"'] & q_2 \arrow[d, "\text{Hopf}"] & q_2\bar{q}_1 \arrow[d, "\text{Hopf}"] \arrow[r] & \cos\frac{\theta}{2} + \sin\frac{\theta}{2}(n_1 i + n_2 j + n_3 k) \\ F_1 & F_2 & F_2 F_1^T \arrow[r] & (\theta, \vec{n}) \end{tikzcd}$$

But this doesn't mean that we can work backwards, since we can only determine θ from "downstairs" information to $\pm 2\pi K$ and we would need to determine it in $[0, 4\pi)$ to reconstruct the quaternion $q_2\bar{q}_1$ unambiguously.

Equivalently

$$\pm q_1 \xrightarrow[\text{Hopf}]{} F_1, \quad \pm q_2 \xrightarrow[\text{Hopf}]{} F_2$$

so we can't know the <u>sign</u> of

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$ bey knowing F_1, F_2.

This means that we have really only proved

$$\det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \pm |q_1||q_2| \sin\frac{\theta_{12}}{2} (n_2 + n_3 i),$$

and hence that there's a <u>choice</u> of signs which make the Plücker relation hold.

Which means the question remains open...