Schubert Cells II

Last time, we introduced the cell corresponding to multiindex $j = (j_1, \dots, j_k)$ as the set

$$C_j = \left\{ Y \in G_k(\mathbb{C}^n) \;\middle|\; Y_{\ell j_\ell} = 1, \text{ with 0's above, below and right} \right\}$$

We will often use

$$a_\ell = j_\ell - \ell, \qquad \lambda_\ell = n - k - a_\ell$$

where a_ℓ = # of nonzero elements to the left of (ℓ, j_ℓ) and λ_ℓ = # of zeros to the <u>right</u> of (ℓ, j_ℓ) minus # of pivots below row ℓ.

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We saw earlier that

$$\dim C_j = \sum j_\ell - \ell$$
$$= \sum a_\ell$$
$$= k(n-k) - \sum \lambda_\ell$$

So the λ_ℓ partition the <u>codimension</u> of C_j. Another definition of C_j is

$$C_j(\mathbb{F}_\bullet) = \{ \Psi \in G_k(\mathbb{C}^n) \mid \dim(\Psi \cap \mathbb{F}_{j_\ell}) = \ell \}.$$

which is obvious when you consider the example $C_{3,7,9}$

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{pmatrix}$$

1d intersection with $\mathbb{E}_3$

2d intersection with $\mathbb{E}_7$

3d intersection with $\mathbb{E}_9$

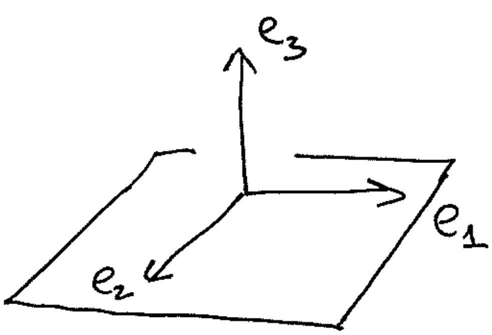
Example. Consider $G_2(\mathbb{R}^3)$.

There are 3 multi-indices of cardinality 2:

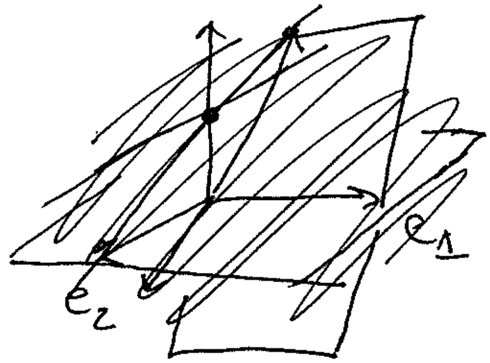
$j = \{1, 2\}$ $\qquad$ $j = \{1, 3\}$ $\qquad$ $j = \{2, 3\}$

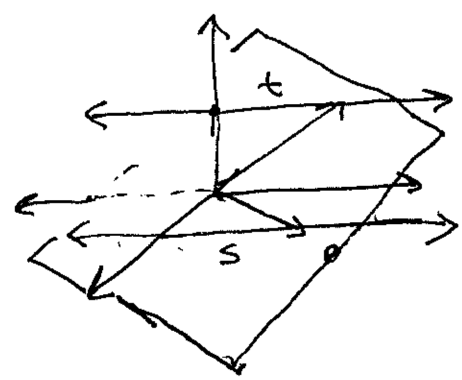
$C_{1,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ $\qquad$ $C_{13} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & 1 \end{pmatrix}$ $\qquad$ $C_{23} = \begin{pmatrix} * & 1 & 0 \\ * & 0 & 1 \end{pmatrix}$

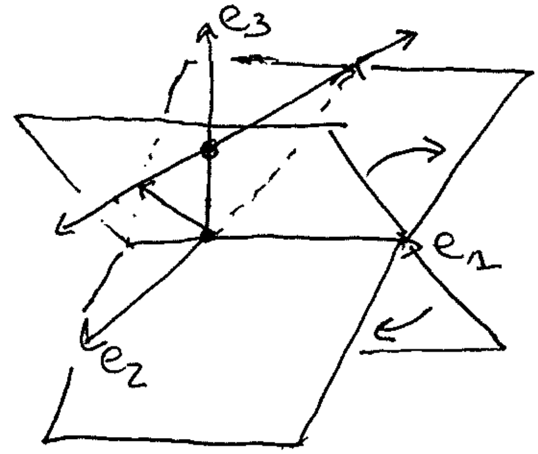
$\dim = 1-1+2-2 = 0$ $\qquad$ $\dim = 1-1+3-2 = 1$ $\qquad$ $\dim = 2-1+3-2 = 2$



one entry







two parameter family with basis vectors $(s, 1, 0)$ and $(t, 0, 1)$

one parameter family containing e_1 and some vector on the line $(0, t, 1)$.

Our goal now is to embed the Grassmannians in higher dimensional spaces so that their extrinsic geometry means something. We first review:

Definition. Given a vector space V, we can construct the k-th tensor power $V \otimes V \otimes \cdots \otimes V$ to be the vector space ~~with basis elements~~ spanned by all k-tensors in the form $v_1 \otimes \cdots \otimes v_k$.

We need to keep track of the fact that <u>a tensor is always an equivalence class</u> up to the tensor relations

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w$$

$$v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

$$cv \otimes w = v \otimes cw = c(v \otimes w).$$

The <u>tensor algebra</u> of V is the direct sum

$$T(V) = T^0 V \oplus T^1 V \oplus \cdots \oplus T^K V \oplus \cdots$$

where the "0-tensors" are just elements of the ~~grou~~ field of scalars for V.

The <u>exterior algebra</u> $\Lambda(V)$ is the quotient of $T(V)$ by the ideal generated by 2-tensors in the form $x \otimes x$.

We can define subspaces $\Lambda^K(V)$ by the tensors of rank K. These are generated by tensors of the form

$$v_{j_1} \otimes \cdots \otimes v_{j_K} \quad \text{for } j = (j_1, \ldots, j_K)$$

and the v_i are any basis for V. (All such tensors with <u>repeated</u> indices are in the ideal generated by $x \otimes x$ and hence equivalent to zero in $\Lambda^K(V)$.)

We denote elements in $\Lambda(V)$ by

$x \wedge y \wedge z \ldots$ instead of $x \otimes y \otimes z \ldots$

to remind ourselves that they are equivalence classes of tensors since $\Lambda(V)$ is a quotient space of $T(V)$.

Now the inner product on V defines an inner product on $\Lambda^k(V)$ given by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = \det\left(\langle v_i, w_j \rangle\right)$$

where $\left(\langle v_i, w_j \rangle\right)$ is the $k \times k$ Gramian matrix of inner products.

Further, the dimension of $\Lambda^k(V)$ is $\binom{n}{k}$ if $\dim V = n$.

Here is a useful fact: if we have any linear map $f: V \to W$ then it induces a unique graded algebra homomorphism given by

$$\Lambda^k(f)(v_1 \wedge \dots \wedge v_k) = f(v_1) \wedge \dots \wedge f(v_k).$$

In particular, if we take

$$\Lambda^n(f) : \Lambda^n(V) \to \Lambda^n(W)$$

this is a mapping from a 1-d vector space to another 1-d space given by $\det f$.

Proposition. If $\mathrm{span}(v_1, \dots, v_k) = \mathrm{span}(v_1', \dots, v_k')$, then $v_1 \wedge \dots \wedge v_k = \lambda\, v_1' \wedge \dots \wedge v_k'$.

Proof. ~~Extend $v_1, \dots, v_k$~~ If $v_1, \dots, v_k$ are linearly dependent, so are $v_1', \dots, v_k'$ and both sides are zero.

If not, we can extend the V_i to a basis for V. There is a linear map $f: V \to V$ which takes the v_i' to the v_i, ~~restrict it to~~ thus expressing each v_i' $(i \in 1, \dots, k)$ as a linear combination of v_i $(i \in 1, \dots, k)$

$$V_1' \wedge \dots \wedge V_k' = \left(\sum f_{1j} v_j\right) \wedge \dots \wedge \left(\sum f_{kj} v_j\right).$$

Using the relation $v_j \wedge v_j = 0$, we can rearrange the right hand side to be

$$= \lambda (v_1 \wedge \dots \wedge v_k)$$

as desired. In fact, if we restrict ~~each of the~~ f to $\mathrm{span}(v_1, \dots, v_k)$, we see that the scalar λ is exactly the determinant of this restriction. $\square$

We can now define a map from $G_k(V)$ to the projective space $P(\Lambda^k V)$ given by

$$p(\mathrm{span}(v_1, \ldots, v_k)) = v_1 \wedge \ldots \wedge v_k.$$

This is called the Plücker map.

Claim. p is well-defined.

True, since changing basis changes image by a scalar, which is an element of the same equivalence class in projective space.

Claim. p is 1-1.

Given some $v_1 \wedge \ldots \wedge v_k$, $\mathrm{span}(v_1, \ldots, v_k)$ is precisely the set of ω so that

$$\omega \wedge v_1 \wedge \cdots \wedge v_k = 0.$$

Claim. An element of $\mathbb{P}(\Lambda^k \mathbb{C}^n)$ is in the image of $p \Leftrightarrow$ the element can be written as $v_1 \wedge \cdots \wedge v_k$ for some $v_1, \ldots, v_k \in \mathbb{C}^n$.

Proof. $\Leftarrow$ is abvious. ~~Parsee~~ so is $\Rightarrow$.

This means that the image of p is a <u>subvariety</u> of $\mathbb{P}(\Lambda^k \mathbb{C}^n)$.