Schubert Calculus and Topology

The Grassmannian $G_k(\mathbb{C}^n)$ can be represented by matrices in $Mat_{k \times n}(\mathbb{C})$ (we usually use columns, but here the span is of <u>row</u> vectors).

Let $j = \{i_1, \dots, i_k\}$ be a multi-index of cardinality k where each $i_\ell \in 1, \dots, n$.
Let

$$Y_{j^\circ} \subset \mathbb{C}^n = \text{Span}\{ e_\ell \mid \ell \ \underline{\text{not}} \ \text{in } j \}.$$

and

$$U_j = \{ \Psi \in G_k(\mathbb{C}^n) \mid \Psi \cap \Psi_{j^\circ} = \{0\} \}.$$

We claim that any $\Psi \in U_j$ has a unique representation in the form

$$\begin{pmatrix} \vert & \vert & \vert & & \vert & & \vert & \vert & \vert & & \vert \\ \vert & \vert & e_1 & \cdots & e_2 & \cdots & e_k & \vert & \vert & \cdots & \vert \\ \vert & \vert & \vert & & \vert & & \vert & \vert & \vert & & \vert \end{pmatrix},$$

elements of j

or so that the submatrix formed by selecting columns in j is the identity matrix.

Proof. We know that any element of U_j has intersection 0 with Y_j°. So suppose the $K \times K$ submatrix given by the columns in j is not full rank. There exist some coefficients so that

$$\Pi_j \left(a_1 (\text{row } 1) + \dots + a_K (\text{row } K) \right) = 0$$

~~be~~ where Π_j is projection to $\mathbb{C}^K$. But then $a_1(\text{row } 1) + \dots + a_K(\text{row } K) \in Y_j^{\circ}$. $\times\!\!\times$

So $\pi_j \Psi$ is full rank in $\mathrm{Mat}_{K\times K}(\mathbb{C})$, and can hence be inverted by a unique $K\times K$ matrix. Apply this matrix to Ψ to get the (unique) representation Ψ_{j°. $\square$

We will think of the set of $K\times n$ representations for Ψ_{j° as an $K\times(n-K)$ dimensional coordinate patch for $G_K(\mathbb{R}^n)$. Clearly, this patch covers all but a measure zero subset of $G_K(\mathbb{R}^n)$ and the set of all such patches covers $G_K(\mathbb{R}^n)$.

Exercise. Prove the transition functions are holomorphic so that $G_K(\mathbb{R}^n)$ is a complex manifold.

Example. $$\underbrace{\begin{pmatrix} 6 & 3 & 0 & 0 \\ 4 & 0 & 2 & 0 \\ 9 & 0 & 1 & 1 \end{pmatrix}}_{\text{reduce to row-echelon form}} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 0 & 1 \end{pmatrix}$$

Now recall that we defined

$\mathbb{C}P^n$ = ~~p~~ equivalence classes of points in $\mathbb{C}^{n+1} - \{\vec{0}\}$ up to scalar multiplication.

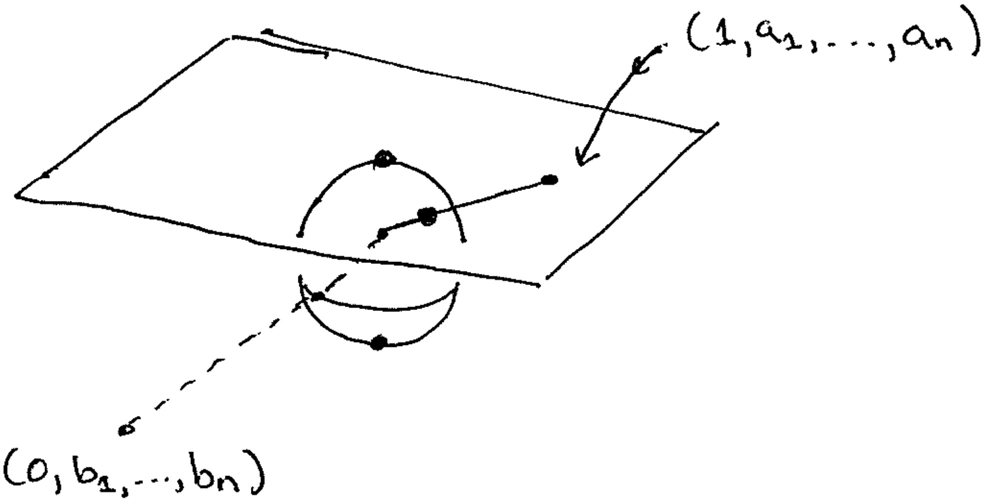
But we can also see

these are only members of eq. classes

$$\mathbb{C}P^n = \begin{cases} (1, a_1, \ldots, a_n), & \text{"ordinary points in } \mathbb{C}^n\text{"} \\ (0, b_1, \ldots, b_n), & \text{"completion points at } \infty\text{"} \end{cases}$$

note that these are members of equivalence classes, consisting ~~of~~ of other completion points.

the picture is



Definition. A projective linear space $L \subset \mathbb{C}P^n$ is the set of points $P = (p_0, \dots, p_n)$ whose coordinates satisfy

$$B \cdot P = 0$$

for some constant matrix B in $Mat_{(n-k)\times(n+1)}(\mathbb{C})$.

We say L is k-dimensional if these $(n-k)$ equations are independent (or B has full rank).

Now the kernel of B depends only on span (rows of B), so we see that

$G_k(\mathbb{C}^n)$ = parameter space for the variety of $k-1$ dimensional planes in $\mathbb{C}P^{n-1}$, or $G_{k-1}(\mathbb{C}P^{n-1})$.

A bit of explanation is in order:

$$Y \in G_K(\mathbb{C}^n) \rightarrow \text{a } K \times n \text{ matrix}$$

the dual (perp) space is an $(n-K) \times n$ matrix. This is a rank $((n-1)-(K-1))$ matrix of dimensions $((n-1)-(K-1)) \times ((n-1)+1)$, which corresponds to a projective $K-1$ space in $\mathbb{C}P^{n-1}$.

Now we are going to define Schubert cells.

Definition. A flag $\mathbb{F}_\bullet$ for an n-dimensional vector space $\mathbb{V}$ is a collection of nested subspaces whose dimensions differ by 1:

$$\mathbb{F}_\bullet : \mathbb{F}_1 \subset \mathbb{F}_2 \subset \ldots \subset \mathbb{F}_n = \mathbb{V}.$$

Example. The standard flag for $\mathbb{C}^n$ would have $\mathbb{E}_i = \operatorname{span}\{e_1, \ldots, e_i\}$.

Let us fix a flag $\mathbb{F}_\bullet$ of $\mathbb{C}^n$. We can construct Schubert cells

$$G_K(\mathbb{C}^n) = \bigsqcup_{j \in [n]} C_j$$

where $[n]$ is the set of multi-indices of cardinality K in $\{1, \ldots, n\}$, and C_j consists of planes whose matrix representation ~~consists of~~ has a 1 in ~~the~~ each (ℓ, j_ℓ) position with zeros above, below, and right for

$$j = (j_1 < j_2 < j_3 < \cdots < j_n).$$

By Gaussian elimination, each $\varphi \in G_K(\mathbb{C}^n)$ lies in exactly one C_j: we say that ~~such~~ a cell C_j represents K-planes which meet the flag ~~at~~ with the same <u>attitude</u>.

For a given $j = \{j_1, \dots, j_k\}$, we see

$$\dim(C_j) = \sum_{\ell=1}^{k} j_\ell - \ell.$$

Example. Consider $G_3(\mathbb{C}^{10})$.

$\begin{pmatrix} \cdots & \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{pmatrix}$ → row reduction trying to make zeros in upper left results in (say)

$$\begin{pmatrix} 6 & 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 7 & 0 & 9 & 3 & 2 & 1 & 0 & 0 & 0 \\ 7 & 5 & 0 & 8 & 4 & 4 & 0 & 3 & 1 & 0 \end{pmatrix}$$ ← pivots are <u>somewhere</u>!

so this is in the $\{3, 7, 9\}$ cell, or has <u>position</u> or <u>attitude</u> $\{3, 7, 9\}$ wrt the standard flag. The remaining elts are in the form

$$\begin{pmatrix} * & * & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 1 & 0 & 0 & 0 \\ * & * & 0 & * & * & * & 0 & * & 1 & 0 \end{pmatrix}$$

with dimension = # of $*$s = 13 = $(3-1)+(7-2)+(9-3)$

We call $j = \{3, 7, 9\}$ the <u>Schubert symbol</u> of the cell.

There are various ways to