Grassmannians and Stiefel Mflds as Quotients of Lie Groups.

Suppose we have an element A of $V_K(\mathbb{R}^n)$.
This is a matrix with K orthonormal columns of length n.

We can complete this matrix to an orthonormal basis for $\mathbb{R}^n$ by adding n-K orthonormal colums all perpendicular to the first K cols.

Observe that $O(n-K)$ acts transitively on the space of such completions of a given A by right-multiplication of matrices in the form $\begin{bmatrix} I & 0 \\ 0 & B_{n-K} \end{bmatrix}$ where $B_{n-K} \in O(n-K)$.

This proves

Proposition. $V_K(\mathbb{R}^n) \cong O(n)/O(n-K)$.

(as long as we recall that by "$O(n-K)$" we mean the <u>specific</u> copy of $O(n-K)$ in $O(n)$ given by the block form above.)

Proof. We can justify the observation by noting that multiplication by $\begin{bmatrix} I & 0 \\ 0 & B_{n-K} \end{bmatrix}$ doesn't change the first K columns at all, the <u>span</u> of the last $n-K$ colums, or the orthonormality of the last $n-K$ columns. □

Similarly, ~~action~~ multiplication by $\begin{bmatrix} B_K & 0 \\ 0 & I \end{bmatrix}$ is transitive on all orthogonal matrices whose <u>first</u> K colums have the same span.

Putting these together, we define

Definition. The Grassmann manifold $G_k(\mathbb{R}^n)$ is the space of k-dimensional linear subspaces of $\mathbb{R}^n$.

We can immediately see:

Proposition.
$$G_k(\mathbb{R}^n) \cong V_k(\mathbb{R}^n)/O(k)$$
$$\cong O(n)/O(k)\times O(n-k)$$
$$\cong V_{n-k}(\mathbb{R}^n)/O(n-k)$$
$$\cong G_{n-k}(\mathbb{R}^n).$$

This shows you the advantage of the Grassmann manifolds immediately: they have a duality property that Stiefel manifolds just don't.

We now want to do some elementary dimension~~s~~ counting. The easy way to count dimensions is to use our product representation:

Lemma. $\dim(V_K(\mathbb{R}^n)) = nK - \frac{K(K+1)}{2}$ (real)

$\dim(V_K(\mathbb{R}^n)) = nK - \frac{K(K+1)}{2}$ (complex).

Proof. Follows directly from the fact that rank K, $n \times K$ matrices are an open set in $n \times K$ matrices and so have dimension nK.

In particular, this recovers

$$\dim(O(n)) = \dim(V_n(\mathbb{R}^n)) = n^2 - \frac{n(n+1)}{2}$$

$$= \cancel{2n}\,\frac{2n^2 - n^2 - n}{2} = \frac{n^2 - n}{2} = \frac{n(n-1)}{2}.$$

Using this we can compute dimensions for the Grassmann manifolds, too.

Lemma. $\dim G_K(\mathbb{R}^n) = K(n-K)$.

Proof. Exercise.

Here is another way to compute the dimensions which is a bit more hands-on.

Definition. A Lie group is a group G which is also a smooth manifold so that multiplication and inverses are smooth maps. The corresponding Lie algebra $\mathfrak{g}$ is the tangent space to the group at the identity.

"Clearly", our matrix groups are Lie groups. We can compute the Lie algebras by differentiation.

Proposition. The Lie algebra $o(n)$ is the space of skew-symmetric matrices.

Proof. Let $A(t)$ be a curve in $O(n)$ so that $A(0) = I$. Consider $A'(0)$. We know

$$A(t)\,A^T(t) = I \quad \text{for all } t,$$

so

$$A'(t)\,A^T(t) + A(t)\,A^{T'}(t) = 0.$$

Evaluating at $t=0$, we get

$$A'(0) + A'(0)^T = 0,$$

or $A'(0)$ is skew-symmetric. $\square$.

Corollary. $\dim O(n) = \dim o(n) = \frac{n(n-1)}{2}$.

This is just the dimension of the skew symmetric matrices!

We can now analyze the Stiefel manifolds as quotient spaces

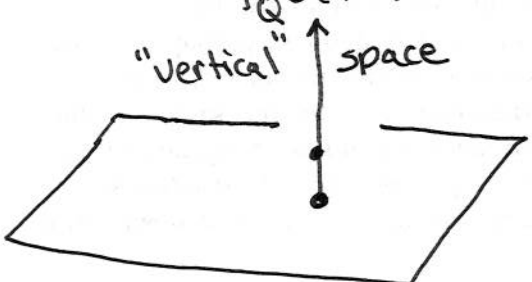
$$\begin{array}{ccc} O(n-k) & \longrightarrow & O(n) \\ & & \downarrow \pi \\ & & V_k(\mathbb{R}^n) \end{array}$$

We want to think about the corresponding tangent space picture.

$T_Q O(n-k)$

"vertical" space

$T_Q O(n)$ is the ambient space



"horizontal space"

$(T_Q O(n-k))^{\perp}$ = copy of the tangent ~~vec~~ space to $V_k(\mathbb{R}^n)$ at $\pi(Q)$.

Now the fiber $[Q]$ is the space of matrices in $O(n)$ in the form

$$[Q] = \left\{ Q \begin{bmatrix} I & 0 \\ 0 & B \end{bmatrix} \middle| \; B \in O(n-K) \right\}$$

which has tangent space equal to

$$\mathrm{Vert}_Q = \left\{ Q \begin{bmatrix} 0 & 0 \\ 0 & C \end{bmatrix} \middle| \; C \text{ is } n{\times}n \text{ skew symmetric} \right\}$$

by our previous calculation.

We can now get the tangent space $T_{[Q]}V_K(\mathbb{R}^n)$ as the orthogonal complement of the vertical space:

$$\mathrm{Horiz}_Q = \left\{ Q \begin{bmatrix} A & -B^* \\ B & 0 \end{bmatrix} \middle| \; \begin{array}{l} A \text{ is } K{\times}K \text{ skew symm.} \\ B \text{ is } \\ (n-K){\times}K, \text{ arbitrary} \end{array} \right\}$$

This lets us recompute

$$\dim V_K(\mathbb{R}^n) = (n-K)\cdot K + \frac{K(K-1)}{2}$$

and it gives us more!

Definition. ~~Ex~~ A map $f : X \to Y$ of Riemannian manifolds is a Riemannian submersion if $df : \underbrace{\ker(df)^{\perp}}_{\text{horizontal space.}} \to TY$ is an isometry.

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Equivalently, we can define an inner product (and hence a metric) on $V_k(\mathbb{R}^n)$ by taking the metric on $O(n)$ restricted to the horizontal space.

The metric on $V_k(\mathbb{R}^n)$ is then the subspace metric on $O(n) \subset Mat_{n \times n}$ restricted to matrices in the form

$$\Delta = Q \begin{pmatrix} A & -B^* \\ B & 0 \end{pmatrix}$$

which is then

$$\langle \Delta_1, \Delta_2 \rangle = \frac{1}{2} \operatorname{tr} \left(\left(Q \begin{pmatrix} A_1 & -B_1^* \\ B_1 & 0 \end{pmatrix} \right)^* \left(Q \begin{pmatrix} A_2 & -B_2^* \\ B_2 & 0 \end{pmatrix} \right) \right)$$

$$= \frac{1}{2} \operatorname{tr} \begin{pmatrix} A_1 & -B_1^* \\ B_1 & 0 \end{pmatrix}^* \begin{pmatrix} A_2 & -B_2^* \\ B_2 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \operatorname{tr} \begin{pmatrix} A_1^* & B_1^* \\ -B_1 & 0 \end{pmatrix} \begin{pmatrix} A_2 & -B_2^* \\ B_2 & 0 \end{pmatrix}$$

$$= \frac{1}{2} \operatorname{tr} \left[\begin{pmatrix} A_1^* A_2 + B_1^* B_2 & -B_1 B_2^* \\ -B_1 A_2 & B_1 B_2^* \end{pmatrix} \right]$$

$$= \frac{1}{2} \operatorname{tr} (A_1^* A_2) + \underbrace{\frac{1}{2} \operatorname{tr} (B_1^* B_2) + \frac{1}{2} \operatorname{tr} (B_1 B_2^*)}_{\text{same by trace laws}}$$

$$= \frac{1}{2} \operatorname{tr} (A_1^* A_2) + \operatorname{tr} (B_1^* B_2).$$

This looks funny (why are the B terms "overweighted"?) until you remember that A is skew-symmetric and so only has "half" its entries independent.

This tells us $O(n) \to V_K(\mathbb{R}^n)$ is a Riemannian submersion with respect to these metrics.

Now let's play the same game with the projection $O(n) \to G_K(\mathbb{R}^n)$.

Here

$$[Q] = \left\{ Q \begin{bmatrix} B_K & 0 \\ 0 & B_{n-K} \end{bmatrix} \,\middle|\, \begin{array}{l} B_K \in O(K) \\ B_{n-K} \in O(n-K) \end{array} \right\}.$$

and so the vertical space is

$$\mathrm{Vert}_Q = \left\{ Q \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} \,\middle|\, \begin{array}{l} A, C \text{ skew-sym} \\ A \text{ is } K \times K,\ C \text{ is } (n-K) \times (n-K) \end{array} \right\}$$

We then have

$$\mathrm{Horiz}_Q = \left\{ Q \begin{bmatrix} 0 & -B^* \\ B & 0 \end{bmatrix} \,\middle|\, B \text{ is } (n-K) \times K \right\}.$$

and clearly $\dim G_K(\mathbb{R}^n) = (n-K) \times K$, and the metric on $G_K(\mathbb{R}^n)$ is given by the following. If

$$\Delta \in \text{Horiz}_Q = Q\begin{bmatrix} 0 & -B^* \\ B & 0 \end{bmatrix}$$

then we have

$$\langle \Delta_1, \Delta_2 \rangle = \operatorname{tr} BB^*,$$

which is pretty sweet! Again,

$$O(n) \to G_k(\mathbb{R}^n)$$

is a submersion. In fact, we conclude that $V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ is a submersion too, since the vertical space is just the "A" part of $V_k(\mathbb{R}^n)$'s tangent space.

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Coda: If $O(n)$ gets its metric from being a subspace of the Euclidean space of $n \times n$ matrices, and $V_k(\mathbb{R}^n)$ gets a ~~quotient of that~~ submersion of that metric, is that the same metric $V_k(\mathbb{R}^n)$ gets as a submanifold of the Euclidean space of $n \times k$ matrices?

No! The submanifold metric $V_k(\mathbb{R}^n) \subset Mat_{n,k}$ does not count the skew-symmetric "A" and arbitrary "B" terms differently.