The Plücker Relations II

We now need to show that the Plücker relations determine the points in ~~$\mathbb{R}$~~ $\mathbb{P}^N$ (recall $N = \binom{n}{k} - 1$) which come from points in $G_k(\mathbb{C}^n)$ through the Plücker embedding.

Proof. A point α in $\mathbb{P}^N$ has at least one nonzero coordinate; let it correspond to the multindex $\ell = (\ell_1, \ldots, \ell_k)$.

If α obeys the Plücker relations, we claim that all $N+1$ coordinates of α are determined by the $k(n-k)+1$ coordinates of the form

$$p(\ell_1 \ldots \hat{\ell}_\lambda \ldots \ell_k j_\beta), \quad j_\beta \in 1, \ldots n$$

~~which are not always~~

To verify this count, recall that

coordinates are labelled by multindices, which cannot contain repeated elements, and don't depend on order, so that if $j_\beta \in l_1, \ldots, l_K$, $j_\beta = l_\lambda$.

There is therefore only <u>one</u> way to have $j_\beta \in l_1, \ldots, l_K$.

If $j_\beta \notin l_1, \ldots, l_K$, it is chosen from $n-K$ possible indices. Since there are K possible choices for λ, there are $K(n-K)$ coordinates in this form. $\square$

Now suppose we have some arbitrary multindex $j = (j_1, \ldots, j_K)$. Exactly m of these are NOT in $l_1, \ldots, l_K$ (for some m).

~~The Plücker relation for~~ $\cancel{j_0,\ldots,j_k}$

Choose some j_β among these m.

The Plücker relation for $j_1,\ldots,\hat{j}_\beta,\ldots,j_k$ and $l_1,\ldots,l_k,j_\beta$ yields

$$p(j_0,\ldots,\hat{j}_\beta,\ldots,j_k,j_\beta)\,p(l_1,\ldots,l_k) =$$
$$\pm\sum_{\lambda=1}^{k} p(j_0,\ldots,\hat{j}_\beta,\ldots,j_k,l_\lambda)\,p(l_1,\ldots,\hat{l}_\lambda,\ldots,l_k,j_\beta).$$

Now if l_λ is one of the $m-1$ $j_0,\ldots,\hat{j}_\beta,\ldots,j_k$ which are among $l_1,\ldots,l_k$, we are to take the value of p as zero and this term vanishes from the relation.

The remaining terms have $j_0,\ldots,\hat{j}_\beta,\ldots,j_k,l_\lambda$ so that exactly $m-1$ are among $l_1,\ldots,l_k$.

The terms $p(l_1,\dots,\hat{l}_\lambda,\dots,l_k,j_\beta)$ have already been fixed, so ~~we may solve for~~ since $p(l_1,\dots,l_k)\neq 0$, we can solve for

$$p(j_0,\dots,j_k) = \pm\, p(j_0,\dots,\hat{j}_\beta,\dots,j_k,j_\beta)$$

in terms of coordinates with $(m-1)$ of the j_i not in $l_1,\dots,l_k$.

This procedure can be iterated until $m=2$ ~~(m=2)~~, at which point we are expressing everything in terms of known coordinates. This proves that our $k(n-k)+1$ known coordinates determine the rest. $\square$

So assume $p(l_1, \dots, l_K) = 1_*$ (wlog).

We will now construct a K-plane in $\mathbb{C}^n$ with these Plücker coordinates. Consider the vectors

$$L = \begin{matrix} \vec{V}_1 = (p(1, l_2, \dots, l_K), p(2, l_2, \dots, l_K), \dots, p(n, l_2, \dots, l_K)) \\ \vdots \\ \vec{V}_K = (p(l_1, \dots, l_{K-1}, 1), p(l_1, \dots, l_{K-1}, 2), \dots, p(l_1, \dots, l_{K-1}, n)). \end{matrix}$$

We claim these span our plane.

a) They are linearly independent.

Consider the column l_λ. Each $p(l_\lambda, l_2, \dots, l_K)$ to $p(l_1, \dots, l_{K-1}, l_\lambda)$ contains a repeated index and so is zero <u>except</u> the λth row, which is ± 1.

This means that the submatrix L_ℓ is an identity matrix (and has det 1, as required). In particular, this shows L has full rank.

b) This plane has correct Plücker coordinates after embedding in $\mathbb{P}^N$.

Suppose ~~ex~~ we have coordinate $j = (\ell_1, \dots, \hat{\ell}_\lambda, \dots, \ell_k, j_\beta)$. Columns $\ell_1, \dots, \hat{\ell}_\lambda, \dots, \ell_k$ of L are columns of the identity matrix. Column j_β is more interesting: only row λ will contribute to the determinant $\det L_j$, and that entry is $p(\ell_1, \dots, \ell_{\lambda-1}, j_\beta, \ell_{\lambda+1}, \dots, \ell_k)$ as desired.

Since these coordinates are correct, all others must be, too: we know that the coordinates of $p(L)$ obey the Plücker relations, and we have shown that under these conditions, coordinates in this form determine all coordinates! $\square$

The last part of this is to show that this is the <u>only</u> plane which has these Plücker coordinates.

Actually, this is not hard. If we have some such L', consider

$$p(L'_\ell) \neq 0.$$

Alter the basis for L' by a $K \times K$ matrix so that $p(L'_\ell) = 1$.