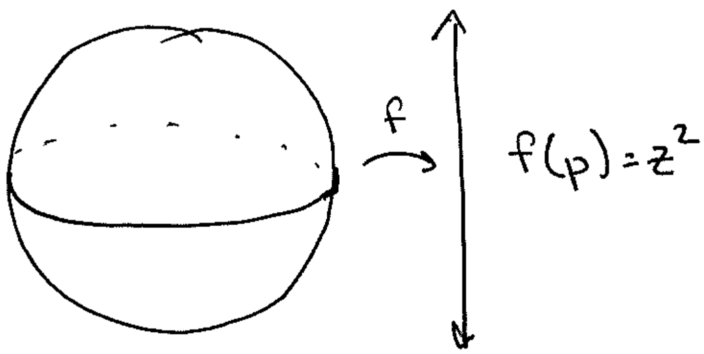
User's Guide to Morse-Bott Theory.

Our basic idea is that we want to generalize Morse theory to cases where critical points form a collection of submanifolds.



For every submanifold $N \subset M$, we can write

$$TM\big|_N = TN_\xi \oplus \nu N_\xi$$

where the operation is Whitney sum of ~~ξ~~ vector bundles and νN_ξ is the normal bundle of $N \subset M$, which is an $n-k$ dimensional vector bundle if $\dim(M) = n$, and $\dim(N) = k$.

(see Math 8230: Characteristic Classes, <u>Vector Bundles</u> and <u>Operations on Vector Bundles</u> for more explanation.)

We now observe that <u>if</u> <u>$f|_N$ is constant</u>, and ~~at~~ f is critical on N, we have that at any $p \in N$,

$$H_{f,p}(\vec{v},\vec{w}) = 0 \text{ if } \vec{v} \in T_pN.$$

That is, ~~the~~ T_pN is an eigenspace of $H_{f,p}(\vec{v},\vec{w})$ and hence $H_{f,p}$ restricts to a quadratic form

$$\mathrm{Hess}^{\nu}_{f,p} : \nu_pN \times \nu_pN \to \mathbb{R}.$$

Definition. If N is a smooth, compact, connected submanifold of M and every point of N is a critical point of f in M, ~~then~~ and if $\mathrm{Hess}^{\nu}_{f,p}$ is nondegenerate at each $p \in N$, then N is a <u>nondegenerate critical submanifold of M</u>.

Along a nondegenerate critical submanifold N, we can split the normal bundle into sub-bundles

$$\nu N = \nu N^+ \oplus \nu N^-$$

along the positive and negative eigenspaces.

Definition. The function f is called a Morse-Bott function if its critical set consists of non-degenerate critical submanifolds N_i. The dimension of νN_i^- is called the Morse index $\lambda(N_i)$ of N_i.

We can now glue together a complex forming M from f. We need the technical hypothesis that f is <u>exhaustive</u> (meaning all sublevel sets are compact).

Theorem (Bott) Let $f: M \to \mathbb{R}$ be an exhaustive Morse-Bott function, and suppose $f^{-1}(c)$ contains finitely critical submanifolds, $N_1, \dots, N_K$. For $i = 1, \dots, K$ let $D^-(N_i)$ be the closed unit disk bundle of the negative normal bundle $\nu^{-}N_i$. Then for small enough ϵ,

$$M^{c+\epsilon} \cong M^{c-\epsilon} \cup_{\partial D^-(N_1)} D^-(N_1) \cup \dots \cup_{\partial D^-(N_K)} D^-(N_K).$$

We can (often) use this structure to understand the homology of M.

Definition. Let F be a field. A Morse-Bott function f is F-orientable if for every critical submanifold the negative normal bundle is F-orientable.

Now we get to the useful part!

Definition. Let F be a field. The F-Morse-Bott polynomial of a Morse-Bott function on a compact manifold M is the polynomial

$$P_f(t) = \sum_N t^{\lambda(N)} P_{N,F}(t)$$

where we sum over critical submanifolds N and $P_{N,F}$ is the Poincaré polynomial of N with coefficients in F.

Theorem (Morse-Bott Inequalities)
Suppose $f: M \to \mathbb{R}$ is an $\mathbb{F}$-orientable Morse-Bott function on a compact manifold M.

$$P_f(t) \succ P_{M,\mathbb{F}}(t)$$

and as before $P_f(-1) = P_{M,\mathbb{F}}(-1) = \chi(M)$, so

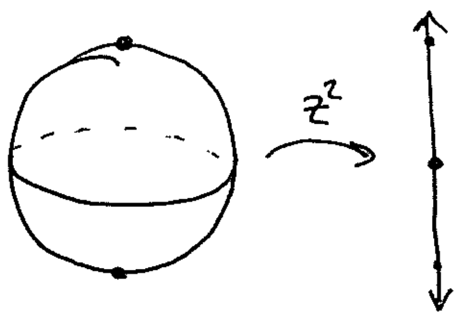
$$\chi(M) = \sum_N (-1)^{\lambda(N)} \chi(N).$$

We now ~~start~~ want a simple criterion for perfect-ness.

Definition. Let $f: M \to \mathbb{R}$ be a Morse-Bott function on a compact manifold M. If for every critical c, and every critical submanifold N, the inclusion $\partial D^-(N) \to M^{c-\epsilon}$ induces the trivial map in homology, then f is called completable.

Theorem. A completable, $\mathbb{F}$-orientable Morse-Bott function on ~~an~~ ~~orient~~ a compact manifold is perfect.

We now check our simple example:



Betti numbers = 1, 0, 1

Poincare polynomial = $t^2 + 1$.

Critical submanifolds:

1) North pole, index = 2, Poincare polynomial = 1.
2) South pole, index = 2, Poincare polynomial = 1.
3) Equator, index = 0, Poincare polynomial = $t+1$

Morse-Bott polynomial: $t^2 + t^2 + t + 1 = 2t^2 + t + 1$.

We see

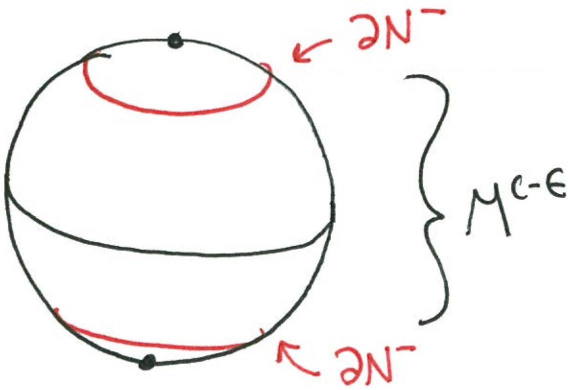
$$2t^2 + t + 1 = t^2 + 1 + (1+t)(t),$$

so

$$P_f(t) \succ P_M(t),$$

as promised.

This is *not* completable; since



~~the two cir~~
the circle ∂N^- is mapping onto the generator in $M^{c-\epsilon} \simeq S^1 \times I$, this inclusion is *not* trivial in homology.

Exercise: Suppose f is an orientable Morse-Bott function for which every critical submanifold has even index and even Poincare polynomial. ~~Then~~ (that is, no odd Betti numbers, not even as a function). Then f is perfect.