A Morse-Bott Function on $G_k(\mathbb{C}^n)$.

We now seek a Morse-Bott function on the Grassmannian $G_k(\mathbb{C}^n)$. We will use the equivalence between subspaces and <u>projectors</u>.

Lemma. There is a smooth embedding of $G_k(\mathbb{C}^n)$ into $\mathrm{Mat}_{n\times n}(\mathbb{C})$ given by associating the orthogonal projection $P_{\mathcal{U}}$ to any subspace $\mathcal{U}$ with $\mathcal{U}$.

Lemma. If the $n\times k$ matrix A is an orthonormal ~~bas~~ basis for $\mathcal{U}$, AA^T is $P_{\mathcal{U}}$.

Lemma. The rank of $P_{\mathcal{U}}$ is $\dim(\mathcal{U})$.

Now suppose we let $\mathbb{C}^n = \mathbb{C}^1 \oplus \mathbb{C}^{n-1}$ (we're really just picking any linear combination of coordinates here, but we'll call it e_0).

Proposition. The function $f : Gr_k(\mathbb{C}^n) \to \mathbb{R}$ given by $f(\mathcal{U}) = \mathrm{Re}\, \mathrm{tr}(P_{e_0} P_{\mathcal{U}})$ is an orientable Morse-Bott function.

Proof. We claim first that

$$f(\mathcal{U}) = \mathrm{Re}\langle P_{\mathcal{U}} e_0, e_0 \rangle.$$

In coordinates, if $u_1, \ldots, u_k$ is an orthonormal basis for $\mathcal{U}$, then

$$P_{\mathcal{U}} = [u_1 \ldots u_k][u_1 \cdots u_k]^* \qquad P_{e_0} = [e_0][e_0]^*$$

$$= \begin{bmatrix} \sum_{\alpha=1}^{k} u_\alpha(i)\, \overline{u_\alpha}(j) \\ \\ \end{bmatrix} \qquad = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{bmatrix}$$

So we get

$$P_{e_0} P_u = \begin{bmatrix} \sum_{\alpha=1}^{K} u_\alpha(1)\overline{u_\alpha(1)}, & \cdots & \sum_{\alpha=1}^{K} u_\alpha(1) u_\alpha(\cancel{k}\, n) \\ 0 \; \cdots \; 0 & & 0 \\ \vdots \qquad \vdots & & \\ 0 \; \cdots \; 0 & & 0 \end{bmatrix}$$

and

$$\operatorname{tr} P_{e_0} P_u = \sum_{\alpha=1}^{K} u_\alpha(1)\overline{u_\alpha(1)}.$$

so

$$\operatorname{Re} \operatorname{tr} P_{e_0} P_u = \operatorname{Re}\left(\sum_{\alpha=1}^{K} u_\alpha(1)\overline{u_\alpha(1)} \right) = \sum_{\alpha=1}^{K} u_\alpha(1)\overline{u_\alpha(1)}$$

Now by comparison, we have

$$P_u e_0 = \begin{bmatrix} \sum_{\alpha=1}^{K} u_\alpha(1)\overline{u_\alpha(1)} \\ \vdots \\ \sum_{\alpha=1}^{K} u_\alpha(1) u_\alpha(n) \end{bmatrix}$$

so

$$\langle P_u e_0, e_0 \rangle = \sum_{\alpha=1}^{K} \cancel{u_\alpha(1)} \, |u_\alpha(1)|^2,$$

which is already real, and equal to above.

We first claim:

Lemma. $0 \le f \le 1$, $f^{-1}(0) = G_k(\mathbb{C}^{n-1})$, $f^{-1}(1) = G_{k-1}(\mathbb{C}^{n-1})$.

Proof. The projection P_u is norm-decreasing, so $\langle P_u e_0, e_0 \rangle \le \langle e_0, e_0 \rangle^2 = 1$. Further, we saw $\langle P_u e_0, e_0 \rangle = \sum_{\alpha=1}^{k} |u_\alpha(1)|^2 \ge 0$.

Now if $f(\mathcal{U}) = 0$, then e_0 lies in $\mathcal{U}^\perp$, and the space of such $\mathcal{U}$ is surely $G_k(\mathbb{C}^{n-1})$. If $f(\mathcal{U}) = 1$, then e_0 lies in $\mathcal{U}$, and there are $k-1$ directions left to choose, all in $\mathbb{C}^{n-1}$, leaving us with $G_{k-1}(\mathbb{C}^{n-1})$ $\square$

Lemma. 0 and 1 are the only critical values.

Proof. Complete $P_u e_0$ to a basis for $\mathcal{U}$, so $P_u e_0 = u_1$, and all $u_2, \dots, u_k$ are normal to e_0 and u_1. We can rotate $\mathcal{U}$ towards e_0 by taking $(\cos\theta\, u_1 + \sin\theta\, e_0, u_2, \dots, u_k)$.

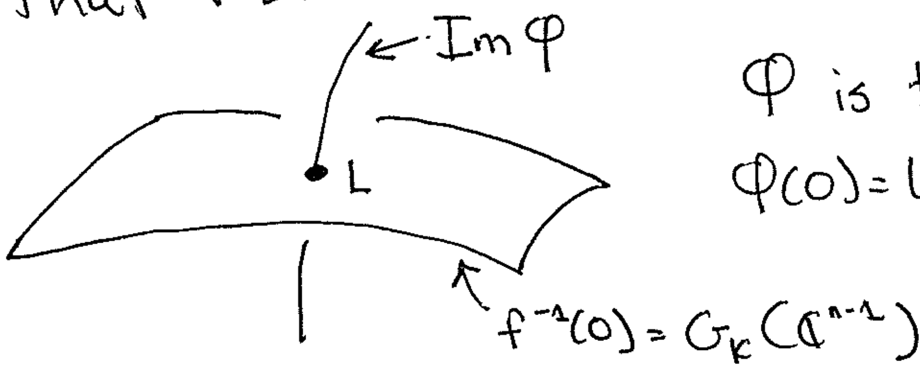
This changes f to first order unless $f = 0$ or 1.

Lemma. $f^{-1}(0)$ and $f^{-1}(1)$ are nondegenerate critical manifolds.

Proof. We know $f^{-1}(0) = G_k(\mathbb{C}^{n-1})$ is a complex submanifold of $G_k(\mathbb{C}^n)$ and thus has complex codimension

$$k(n-k) - k(n-1-k) = k$$

To prove that $f^{-1}(0)$ is nondegenerate, we have to find a $\varphi : \mathbb{C}^k \to G_k(\mathbb{C}^n)$ so that $\forall\, L \in f^{-1}(0)$:



φ is transverse to $f^{-1}(0)$

$\varphi(0) = L$

and $f \circ \varphi$ has a nondeg. local min at 0

We now pick $u \in \mathbb{C}^{n-1}$ and define an operator $X_u : \mathbb{C}^n \to \mathbb{C}^n$ by

$$X_u(e_0) = u, \quad X_u(v) = \langle -v, u \rangle e_0 \text{ for } v \in \mathbb{C}^{n-1}$$

Now we can check for $v, \omega \in \mathbb{C}^{n-1}$

$$\langle X_u(v), \omega \rangle = \langle -v, u \rangle \langle e_0, \omega \rangle_a = 0.$$

$$\langle v, X_u(\omega) \rangle = \langle v, \langle -\omega, u \rangle e_0 \rangle$$
$$= \langle -\omega, u \rangle \langle v, e_0 \rangle = 0.$$

and for e_0,

$$\langle X_u(e_0), e_0 \rangle = \langle u, e_0 \rangle = 0$$

so that if we extend by linearity, let

$$v = a e_0 + v_0$$
$$w = b e_0 + \omega_0$$

then

$$\langle X_u(v), \omega \rangle = \langle a u - \langle v_0, u \rangle e_0, b e_0 + \omega_0 \rangle$$

$$\langle v, X_u(\omega) \rangle = \langle a e_0 + v_0, b u - \langle \omega_0, u \rangle e_0 \rangle$$

$$\langle X_u(v), w\rangle = a\langle u, w_0\rangle - \langle v_0, u\rangle b$$

$$\langle v, X_u(w)\rangle = -\langle w_0, u\rangle a + b\langle v_0, u\rangle,$$

which is to say that X_u is skew-Hermitian. Now we know that any skew-Hermitian matrix defines a 1-parameter family of unitary matrices by $e^{tX_u} : \mathbb{C}^n \to \mathbb{C}^n$. We let

$$\varphi(u) = e^{X_u} L,$$

and define projections $P(u) := P_{\varphi(u)}$ to the subspaces $\varphi(u)$, parametrized by u.

~~Our first claim is that $\varphi(u)$ is transverse to $f^{-1}(0)$ at L, but this requires two stages.~~

Since L itself is a K-dimensional subspace of $\mathbb{C}^{n-1}$, we can restrict

$$\varphi : L \to G_K(\mathbb{C}^n),$$

to get a map $\varphi : \mathbb{C}^K \to G_K(\mathbb{C}^n)$.

Of course, $\varphi(0) = e^{X_0} L = L$, which is a start. We next show that ~~$D\varphi$ has no Kernel.~~ $D\varphi$ has no Kernel. It is convenient to do this in projector coordinates. So consider

$$\begin{aligned} P(u) &= \varphi(u)\varphi(u)^* \\ &= e^{X_u} L L^* (e^{X_u})^* \\ &= e^{X_u} P_L e^{-X_u}, \quad \text{using } (e^{X_u})^* = (e^{X_u})^{-1}. \end{aligned}$$

and compute the directional derivative at 0 of P in the u-direction by taking

$$\frac{d}{dt} P(tu)\Big|_{t=0} = \frac{d}{dt}\left(e^{X_{tu}} P_L e^{-X_{tu}}\right)\Big|_{t=0}.$$

now the map $u \mapsto X_u$ is ~~linear~~ ($\mathbb{R}$)-linear in u, so this is

$$= \frac{d}{dt}\left(e^{tX_u} P_L e^{-tX_u}\right)\Big|_{t=0}$$

$$= X_u e^{tX_u} P_L e^{-tX_u} - e^{tX_u} P_L e^{-tX_u} X_u \Big|_{t=0}$$

$$= X_u P_L - P_L X_u = [X_u, P_L]$$

Lie bracket.

We claim that this is nonzero when $u \neq 0$.

$$\langle (D_u P(0)) e_0, u \rangle = \langle (X_u P_L - P_L X_u) e_0, u \rangle$$

$$= \langle -u, u \rangle = -|u|^2 \neq 0.$$

This proves that $D\varphi(0)$ is injective.
Now let's consider

$$
\begin{aligned}
f(\varphi(u)) &= \cancel{\mathrm{Re}\,A\,\ldots} \\
&= \langle P_{\varphi(u)} e_0, e_0 \rangle \\
&= \langle P(u) e_0, e_0 \rangle \\
&= \langle e^{X_u} P_L e^{-X_u} e_0, e_0 \rangle
\end{aligned}
$$

or, using the fact that e^{X_u} is unitary,

$$
\begin{aligned}
&= \langle P_L e^{-X_u} e_0, e^{-X_u} e_0 \rangle \\
&= \langle P_L (1 - X_u + \tfrac{1}{2} X_u^2 - \tfrac{1}{3} X_u^3 + \ldots) e_0, \\
&\qquad (1 - X_u + \tfrac{1}{2} X_u^2 + \ldots) e_0 \rangle \\
&= \langle \cancel{P_L e_0} - P_L X_u e_0 + \ldots, e_0 - \cancel{X_u e_0} + \ldots \rangle \\
&= -\langle \underbrace{P_L X_u e_0}_{\text{in } L}, \underbrace{e_0}_{\perp \text{ to } L} \rangle + \langle P_L X_u e_0, X_u e_0 \rangle - \ldots \\
&= \langle P_L X_u e_0, X_u e_0 \rangle + \ldots \\
&= \langle P_L u, u \rangle + \ldots = \langle u, u \rangle = |u|^2 + \ldots
\end{aligned}
$$

and

$$f(\varphi(tu)) = $$
$$= t^2|u|^2 + \underbrace{\cdots}_{\text{higher order in } t.}$$

This means that this crit. submfld. is nondegenerate (and transverse as well ~~or~~ since $\nabla f = 0$ at this point). Of course, it has index 0 since the value is a global min! □

Now we check $f^{-1}(1) = G_{k-1}(\mathbb{C}^{n-1})$.
At this point, the strategy is pretty clear: ~~use~~ at each $L \in f^{-1}(1)$, we want to construct a φ map. Now the $\mathbb{C}$-codimension of $f^{-1}(1)$ is

$$k(n-k) - (k-1)(n-1-(k-1)) =$$
$$\cancel{= kn - k^2 - (kn - k^2 + k - n + k - 1)}$$
$$\cancel{= n - 2k + 1}.\quad k(n-k) - (k-1)(n-k)$$
$$= n-k$$

Now if we take some

$$L \in f^{-1}(1),$$ it lies in $G_k(\mathbb{C}^n)$

as always, but we can identify it with

$$L \cap \mathbb{C}^{n-1},$$ a $k-1$ dimensional subspace of $\mathbb{C}^{n-1}$

We see that

$$\dim\left((L \cap \mathbb{C}^{n-1})^{\perp} \right) = n-1-(k-1) = n-k,$$

which is exactly cod $f^{-1}(1)$.

As before, we let $L_0' = (L \cap \mathbb{C}^{n-1})^{\perp}$ and write

$$\Phi : L_0' \to G_k(V), \quad \Phi(u) = e^{X_u} L.$$

The calculation that follows is an exercise, which shows that $f^{-1}(1)$ is a nondegenerate critical manifold of index $2(n-k)$.

We are now ready to compute!

Proposition. Let $P_{K,n}(t)$ be the Poincaré polynomial of $G_K(\mathbb{C}^n)$. Then the odd Betti numbers of $P_{K,n}$ are trivial and

$$P_{K,n}(t) = P_{K,n-1}(t) + t^{2(n-K)} P_{K-1,n-1}(t).$$

Proof. We proceed by induction on $K+n$. When $K+n=5$, we have $K=2$, $n=3$,

$$P_{2,3}(t) = P_{2,2}(t) + t^{2(1)} P_{1,2}(t)$$

or

$$P_{\mathbb{C}P^2}(t) = 1 + t^2 P_{\mathbb{C}P^1}(t)$$

which follows from our previous computation that

$$P_{\mathbb{C}P^n}(t) = 1 + t^2 + t^4 + \dots + t^{2n}$$

Now we know by induction that the Poincaré polynomials of $G_{k}(\mathbb{C}^{n-1})$ and $G_{k-1}(\mathbb{C}^{n-1})$ are even and hence equal to their Morse-Bott polynomials, so f is still perfect on $G_k(\mathbb{C}^n)$, and

$$P_{G_k(\mathbb{C}^n)} = P_{G_k(\mathbb{C}^{n-1})} + t^{2(n-k)} P_{G_{k-1}(\mathbb{C}^{n-1})},$$

by Morse-Bott theory! $\square$.

Now we can solve the recurrence to deduce that

$$P_{G_k(\mathbb{C}^n)} = \frac{\prod_{i=1}^{n}(1-t^{2i})}{\prod_{j=1}^{k}(1-t^{2j})\prod_{i=1}^{n-k}(1-t^{2i})}.$$