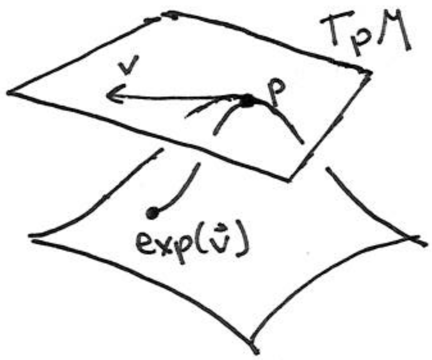
Lie Groups, Lie Algebras and Exp.

In differential geometry, the following construction is standard: Given a vector $\vec{v}$ in T_pM, define $\exp(\vec{v})$ to be the endpoint of the geodesic of length $|\vec{v}|$ starting from p and going ~~length~~ in the $\vec{v}/|\vec{v}|$ direction.



We call this the "exponential map," but why? And how does it work on $V_k(\mathbb{R}^n)$ and $G_k(\mathbb{R}^n)$ in particular? We will now cover some background material which will help us derive an explicit formula for exp.

Definition. Given a real or complex $n \times n$ matrix, ~~A~~ A, we define

$$e^A = I_n + \sum_{p \geq 1} \frac{A^p}{p!}, \text{ or } e^A = \sum_{p \geq 0} \frac{A^p}{p!}$$

if we're willing to take $A^0 = I_n$.

Lemma. This series converges absolutely.

Example. Consider the matrix $A = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$.

We observe that

$$A^1 = \theta \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad A^2 = \theta^2 \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = \theta^2 \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$:= \theta J \qquad\qquad = -\theta^2 I_2.$$

This means

$$e^A = I_2 + \frac{\theta}{1!} J - \frac{\theta^2}{2!} I_2 - \frac{\theta^3}{3!} J + \frac{\theta^4}{4!} I_2 + \frac{\theta^5}{5!} J - \cdots$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) I_2 + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) J$$

$$= \cos\theta \, I_2 + \sin\theta \, J = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

This is a general fact:

The matrix exponential of any skew-symmetric matrix is an orthogonal (unitary) matrix. (Further, this map is surjective onto $SO(n)$, but we'll cover this soon.)

We now develop some of the theory of the matrix exponential.

Lemma. If A is any matrix and U invertible, we have $e^{UAU^{-1}} = Ue^{A}U^{-1}$.

Proof. $(UAU^{-1})^p = UA^pU^{-1}$, so

$$\sum \frac{(UAU^{-1})^p}{p!} = \sum U \frac{A^p}{p!} U^{-1} = U\left(\sum \frac{A^p}{p!}\right)U^{-1}.$$

We also recall

Schur Decomposition Theorem. Any complex matrix A can be written $A = UTU^*$, where T is upper triangular.

Further, the eigenvalues of A are the diagonal entries in T.

Using these, we can show:

Proposition. Given any complex matrix A, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then $e^{\lambda_1}, \ldots, e^{\lambda_n}$ are the eigenvalues of e^A. Further, if $\vec{u}$ is an eigenvector of A, $\vec{u}$ is also an eigenvector of e^A, with corresponding eigenvalue.

Proof. We know $A = UTU^*$, so by our Lemma,

$$e^A = e^{UTU^*} = e^{UTU^{-1}} = U e^T U^{-1},$$

so e^A and e^T have the same eigenvalues. Now taking powers of ~~an~~ the upper triangular matrix T yields matrices $T^2, T^3, \ldots$ which are all upper triangular and have diagonal entries given by T_{ii}^2, T_{ii}^3, and so forth. In particular, this shows that the diagonal entries of e^T are $e^{\lambda_1}, \ldots, e^{\lambda_n}$ where $\lambda_1, \ldots, \lambda_n$ are the diagonal entries of T. Further, e^T is upper triangular, so these are the eigenvalues of e^T (and of T).

To see that any eigen<u>vector</u> of A is an eigenvector of e^A, note that if $A\vec{u} = \lambda\vec{u}$, then $A^n\vec{u} = \lambda^n\vec{u}$ for all n, so $e^A\vec{u} = e^\lambda\vec{u}$. $\square$.

We can show as a consequence that

$$\det(e^A) = \det(e^T) = e^{\lambda_1} \cdots e^{\lambda_n} = e^{\lambda_1 + \cdots + \lambda_n} = e^{\operatorname{tr} A},$$

and as a consequence that e^A is always invertible.

We might guess immediately that

$$(e^A)^{-1} = e^{-A} \quad \text{(it \underline{looks} good)}$$

We need:

Lemma. Given $n \times n$ matrices A, B <u>which commute</u>, $e^{A+B} = e^A e^B$.

Since A and $-A$ commute, we're done.

Now we can define a map from the Lie <u>algebra</u> $so(n)$ of skew-symmetric matrices to the Lie <u>group</u> $SO(n)$.

~~orthogonal~~

Theorem. $\exp : so(n) \to SO(n)$ is well defined and surjective.

Proof. We start with $(e^A)^T = e^{A^T}$ (exercise!),
Now if A is skew-symmetric, we want to prove e^A is in $SO(n)$.

We first check

$$(e^A)^T = e^{(A^T)} = e^{-A} = (e^A)^{-1},$$

as desired. Further $\det(e^A) = e^{\operatorname{tr} A} = e^0 = 1$, showing e^A in $SO(n)$.

Now we need to show exp is surjective, and we'll need two useful linear algebra facts.

1. If A is skew-symmetric, $\exists$ an orthogonal matrix P so that $A = PDP^T$ where

$$D = \begin{bmatrix} D_1 & & 0 \\ & \ddots & \\ 0 & & D_p \end{bmatrix}$$

and each block D_i is either 0 or $\begin{bmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{bmatrix}$ for some θ_i.

2. If $A \in SO(n)$, then $\exists$ an orthogonal matrix P so that $A = PEP^T$ where E is block-diagonal

$$E = \begin{bmatrix} E_1 & & 0 \\ & \ddots & \\ 0 & & E_p \end{bmatrix}$$

and each block is either I or a matrix in the form $E_i = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

(These are basically specializations of Jordan form.)

Now associate D with E block by block in the obvious way, and observe

$$e^A = e^{PDP^{-1}} = P e^D P^{-1},$$

Now powers of a block diagonal matrix are just ~~bo~~ block diagonal with powers of the blocks, so

$$e^D = \begin{bmatrix} e^{D_1} & & \\ & \ddots & \\ & & e^{D}_{p} \end{bmatrix} = \begin{bmatrix} E_1 & & \\ & \ddots & \\ & & E_p \end{bmatrix} = E$$

Since we already saw

$$e^{\begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix}} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

and that $e^0 = I$.

Thus

$$e^A = Pe^D P^{-1} = PEP^{-1} = B,$$

as desired. $\square$

Example. Suppose that $\exp: so(3) \to SO(3)$.
We can write the formula explicitly:
Let

$$A = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \quad \theta = \sqrt{a^2+b^2+c^2}, \quad B = \begin{pmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{pmatrix}.$$

Claim.

$$e^A = \cos\theta\, I + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} B$$

$$= I + \frac{\sin\theta}{\theta} A + \frac{(1-\cos\theta)}{\theta^2} A^2 \quad \text{if } \theta \neq 0.$$

The proof is an exercise. □.

In fact, this represents rotation around the axis (a,b,c) by angle θ!

Theorem. For a matrix Lie group, the matrix exponential $\exp: \mathfrak{g} \to G$ <u>is</u> the Riemannian exponential map $\exp: T_eG \to G$.

Example. Using these facts, we see that lines in $\mathfrak{so}(3)$ = skew-symmetric matrices map to rotations around a fixed axis in $SO(3)$, which are (hence) the geodesics of $SO(3)$.