Grassmannians and Stiefel Manifolds.

We begin with matrices. As we know,

$GL(n,\mathbb{R})$ - invertible matrices $(n\times n)$

$SL(n,\mathbb{R})$ - matrices with determinant 1 $(n\times n)$

$O(n)$ - matrices with $AA^T = I$ $(n\times n)$

$SO(n)$ - matrices with $AA^T = I$ and det $+1$ $(n\times n)$

form a nested collection of groups, and of smooth manifolds:

$$\begin{tikzcd} & O(n) \arrow[dr, hook] & \\ SO(n) \arrow[ur, hook] \arrow[dr, hook] & & GL(n,\mathbb{R}) \\ & SL(n,\mathbb{R}) \arrow[ur, hook] & \end{tikzcd}$$

The orthogonal matrices have rows and columns forming orthonormal bases for $\mathbb{R}^n$, under the usual dot product.

If we recall the Hermitian dot product for complex vectors in $\mathbb{C}^n$,

$$\langle \vec{u}, \vec{v} \rangle = \sum_i u_i \overline{v_i}$$

we can define corresponding complex matrix groups

$$\begin{tikzcd}
 & U(n) \arrow[dr, hook] & \\
SU(n) \arrow[ur, hook] \arrow[dr, hook] & & GL(n,\mathbb{C}) \\
 & SL(n,\mathbb{C}) \arrow[ur, hook] &
\end{tikzcd}$$

where $SU(n)$ and $U(n)$ are called <u>unitary</u> groups. Again, unitary matrices correspond to (Hermitian) orthonormal bases, this time for $\mathbb{C}^n$.

The situation is essentially the same over the quaternions, but the * ~~specia~~ groups of bases are <u>symplectic</u> groups.

We can think of $O(n)$ as a sort of compact core of the (open) and much "floppier" space $GL(n;\mathbb{R})$.

Proposition. $GL(n,\mathbb{R})$ deformation retracts onto $O(n)$.

Proof. The Gram-Schmidt process for orthonormalizing a basis is clearly continuous, preserves orthogonal matrices and can be done continuously. $\square$

In fact, more is true, and we'll want to prove it. We'll need a cool idea from linear algebra. Recall that complex numbers can be written in the form

$$a + bi = re^{i\theta} \leftarrow \text{"polar form"}.$$

where $e^{i\theta}$ is a unit complex # corresponding to a rotation in the plane and r is a scaling. In fact, this generalizes to matrices.

Theorem (Polar decomposition, Gallier p.36 2.1.3, Halmos sec 83)

For any real $n \times n$ invertible matrix A, there is a unique orthogonal matrix R and positive-definite symmetric matrix S, so that

$$A = RS$$

The same is true for complex invertible matrices, but R is unitary instead of orthogonal.

We also need one more idea.

A positive-definite matrix, like a positive number, has a kind of "square root".

Theorem (Cholesky decomposition, Demmel)
Any symmetric positive definite matrix S can be written uniquely as $S = LL^*$ where L is a lower-triangular matrix with strictly positive diagonal entries.

This is true for both real and complex valued S, but L is real valued $\Leftrightarrow$ S is.
Combining these we see

Theorem. Up to homeomorphism,

$$GL(n,\mathbb{R}) = O(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$$
$$GL(n,\mathbb{C}) = U(n) \times \mathbb{C}^{\frac{n(n+1)}{2}}$$

The proof is kind of obvious at this

point, with the observations that

$\frac{n(n+1)}{2}$ = # of nonzero elements in an $n \times n$ lower triangular matrix and $(0,\infty) \simeq (-\infty,\infty)$ up to homeomorphism.

This tells us that $O(n)$ and $U(n)$ really are rigid, in the sense that we can define a projection from $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$ and understand the fiber.

Now if n-frames in $\mathbb{R}^n$ are special, what about k-frames? Here we meet

Definition. The Stiefel manifold $V_k(\mathbb{R}^n)$ or $V_k(\mathbb{C}^n)$ is the manifold of orthonormal (or Hermitian orthonormal) k-frames in $\mathbb{R}^n$ or $\mathbb{C}^n$.

~~We can generalize the polar decomposition to this case. by writing.~~

Clearly, $V_n(\mathbb{R}^n) = O(n)$, $V_n(\mathbb{C}^n) = U(n)$, so we understand these in depth. But what about $V_k(\mathbb{R}^n)$ and $V_k(\mathbb{C}^n)$?

Lemma. $V_k(\mathbb{R}^n) \cong \{ A \in n \times k \text{ matrices} \mid A^T A = I_k \}$

Proof. $A^T A$ is the Gram matrix of the columns: the matrix of dot products. It is $= I_k \iff$ the colums are an orthonormal k-frame. $\square$

A similar statement holds for $V_k(\mathbb{C}^n)$. We then have:

Theorem. For every ~~rank k~~ $n \times k$ matrix A of full rank (that is, rank k), there are ~~is a~~ unique matrices $R \in V_k(\mathbb{R}^n)$, S in symmetric positive definite $k \times k$ matrices so $A = RS$.

Using the Cholesky decomposition as before, we get that

Theorem. Up to homeomorphism,

$$\text{rank } k,\ n \times k \text{ matrices} = \begin{matrix} V_k(\mathbb{R}^n) \times \mathbb{R}^{\frac{k(k+1)}{2}} \\ V_k(\mathbb{C}^n) \times \mathbb{C}^{\frac{k(k+1)}{2}} \end{matrix}$$

depending on whether they are real or complex.

This gives us some immediate insight into the structure of the Stiefel manifolds: they are essentially determined by linear independence of their vectors, not by orthonormality, at least in a very strong topological sense.

Further, this gives us our first numerical insight into the Stiefel manifolds.

Definition. If A is an $n \times m$ matrix, let the Frobenius norm of A be defined by $\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2} = \sqrt{\operatorname{trace}(A^*A)}$.

Theorem. Given an ~~non~~ $n \times K$ matrix A of rank K, the nearest matrix (in the Frobenius norm) to A is the matrix R in $V_K(\mathbb{R}^n)$ given by the polar decomposition.

The same statement holds for $V_K(\mathbb{C}^n)$. This means that we have a natural numerical method for correcting any errors which creep into our orthonormal frames during a computation: compute the polar to project back!