Homotopy Groups of G and S Manifolds.

Another interesting homotopy statement comes from the bundles

$$V_{k-1}(\mathbb{R}^{n-1}) \to V_k(\mathbb{R}^n) \xrightarrow{p} S^{n-1}$$

where p sends a frame $v_1, \dots, v_n$ to its first vector v_1.

Exercise: Prove this is a fiber bundle.

Since $\pi_i(S^{n-1}) = 0$ for $i < n-1$, we have

$$\underbrace{\pi_i(S^{n-1})}_{0} \to \pi_{i-1}(V_{k-1}(\mathbb{R}^{n-1})) \to \pi_{i-1}(V_k(\mathbb{R}^n)) \to \underbrace{\pi_{i-1}(S^{n-1})}_{0}$$

and for $i < n-1$, we have

$$\pi_{i-1}(V_{k-1}(\mathbb{R}^{n-1})) \cong \pi_{i-1}(V_k(\mathbb{R}^n)).$$

We can iterate this process all the way down to

$$\cong \pi_{i-1}(V_2(\mathbb{R}^{n-(k-2)})) \cong \pi_{i-1}(V_1(\mathbb{R}^{n-(k-1)}))$$

$$\cong \pi_{i-1}(S^{n-(k-1)-1}) = \pi_{i-1}(S^{n-k}$$

and conclude that if $i < n-(k-2)-1$, or $i < n-k+1$, ~~we have~~ or $i-1 < n-k$,

$$\pi_{i-1}(V_k(\mathbb{R}^n)) \cong \pi_{i-1}(S^{n-k}) \cong 0,$$

~~so that~~ or, more readably,

$$\pi_j(V_k(\mathbb{R}^n)) \cong 0 \quad \text{for } j < n-k.$$

Similarly,

$$\pi_j(V_k(\mathbb{C}^n)) \cong 0 \quad \text{for } j < 2n-2k+1$$

and

$$\pi_j(V_k(\mathbb{H}^n)) \cong 0 \quad \text{for } j < 4n-4k+3$$

We can now go back to our bundles

$$O(k) \to V_k(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$$

and deduce that

$$\underbrace{\cancel{\pi_j(V_k(\mathbb{R}^n))}}_{0} \to \pi_j(G_k(\mathbb{R}^n)) \to \pi_{j-1}(O(k)) \to \underbrace{\cancel{\pi_{j-1}(V_k(\mathbb{R}^n))}}_{0}$$

implies that for $j < n-k$,

$$\pi_j(G_k(\mathbb{R}^n)) \cong \pi_{j-1}(O(k)).$$

As before, the more interesting case is probably

$$\to \pi_j(V_k(\mathbb{C}^n)) \to \pi_j(G_k(\mathbb{C}^n)) \to \pi_{j-1}(U(k)) \to \pi_{j-1}(V_k(\mathbb{C}^n)) \to$$

which shows that for $j < 2n-2k+1$,

$$\pi_j(G_k(\mathbb{C}^n)) \cong \pi_{j-1}(U(k)).$$

Last time, there were some questions about in what sense $G_2(\mathbb{C}^\infty)$ "was" space curves. (I agree this needs to be clarified!) But this tells us that polygon spaces have plenty of interesting topology (for space polygons) and almost no topology (for plane polygons).

The fact that so many homotopy groups of the Stiefel manifolds vanish helps us compute their homology groups.

~~Theorem (Hatch~~

Hurewicz Theorem (Homotopy-Homology Ladder)

If $\pi_i(X) = 0$ for $0 \leq i < n$, then the (reduced) homology groups $\tilde{H}_i(X)$ vanish for $0 \leq i < n$ and $\pi_n(X) \cong \tilde{H}_n(X;\mathbb{Z})$.

Thus we know that

zeros

$$H_0(V_k(\mathbb{R}^n)) = \mathbb{Z}, \ldots, H_{n-k-1}(V_k(\mathbb{R}^n)) = 0,$$

$$H_{n-k}(V_k(\mathbb{R}^n)) = \pi_{n-k}(V_k(\mathbb{R}^n))$$

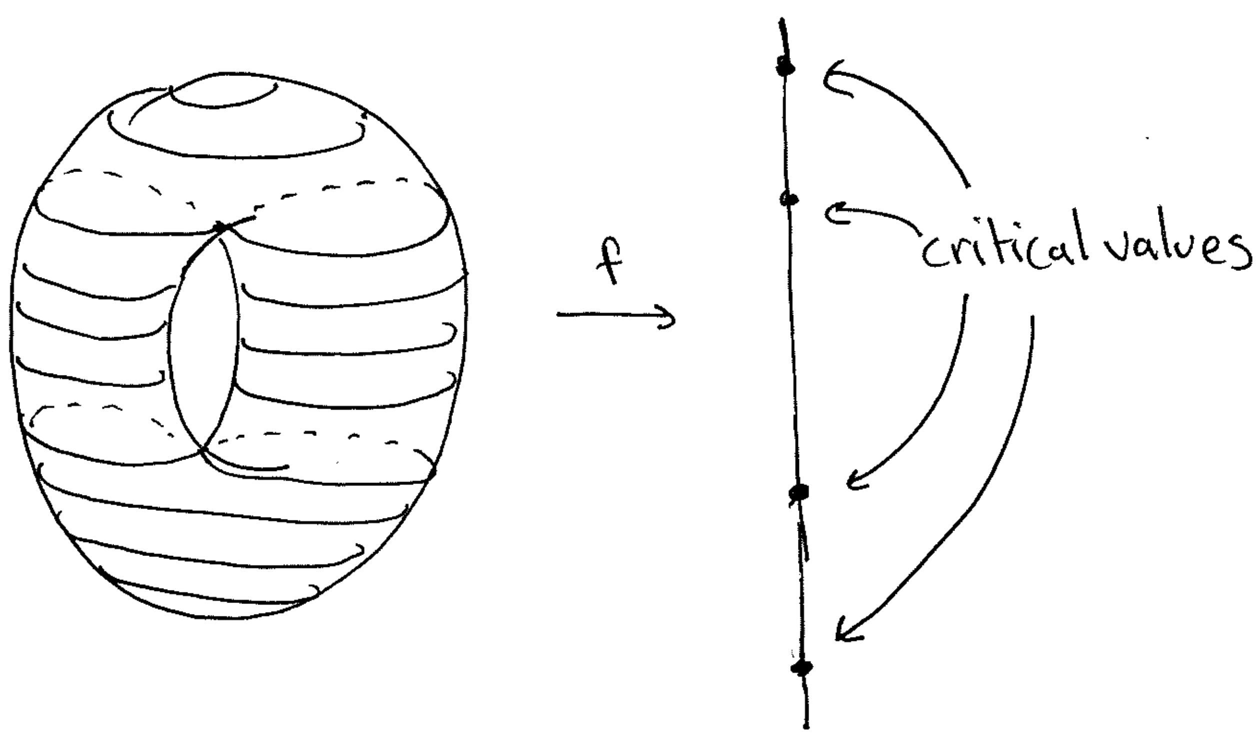
$$H_0(V_k(\mathbb{C}^n)) = \mathbb{Z}, \text{ (zeros)}, H_{2n-2k}(V_k(\mathbb{C}^n)) = 0,$$

$$H_{2n-2k+1}(V_k(\mathbb{C}^n)) = \pi_{2n-2k+1}(V_k(\mathbb{C}^n)).$$

But what more can we say?

We will shortly introduce an explicit cell decomposition on these manifolds which makes the cohomology ring clear (or at least calcolatable!). But for now we will give an approach via Morse-Bott theory.

First, we recall the setup of classical Morse theory. Given a manifold M, we consider functions $f: M \to \mathbb{R}$.



The basic idea is that we can construct a particular CW complex structure on M by following trajectories of ∇f to critical points. We can then compute cellular homology using this CW structure. We need a technical hypothesis:

Definition. Let p_0 be a critical point of the smooth function $f: M \to \mathbb{R}$. The Hessian of f at p_0 is a quadratic form on $T_{p_0}M$ given by

$$\mathrm{Hess}_{f,p_0}(\vec{v},\vec{w}) = XYf(p_0),$$

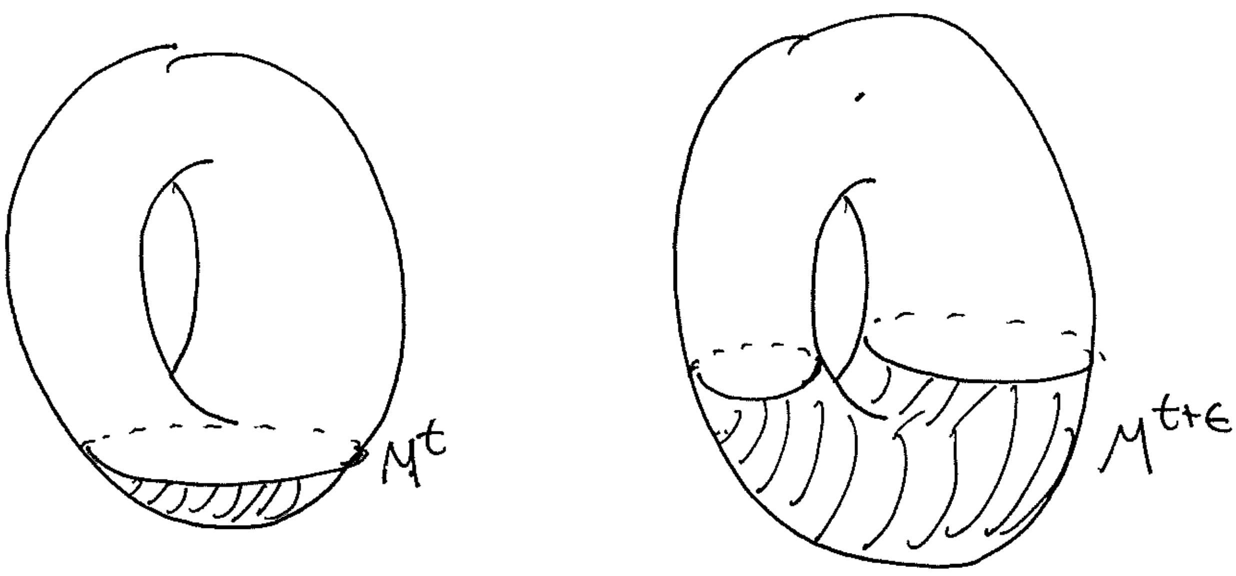
where X and Y are any smooth vector fields extending $\vec{v}, \vec{w}$ and Xf is the derivative of f along the orbit of X as usual.

The Hessian is <u>non-degenerate</u> if

$$\mathrm{Hess}_{f,p_0}(\vec{v},\vec{\omega}) = 0 \text{ for all } \vec{\omega} \Rightarrow \vec{v} = 0.$$

A smooth function is a Morse function if ~~not~~ the Hessian is nondegenerate at each critical point.

The <u>index</u> of a nondegenerate critical point is the number of negative eigenvalues of the Hessian.



Consider the sublevel sets

$$M^t = \{ p \in M \mid f(p) \leq t \}.$$

We see that the topology of M^t changes only when t passes a critical point.

Theorem (weak and strong Morse principles)

If $f^{-1}[a,b]$ contains no critical points, then M^a is diffeomorphic to M^b and M^a is a deformation retract of M^b.

If $f^{-1}[a,b]$ contains a single nondegenerate critical point of index λ, then M^b has the homotopy type of M^a with a λ-cell attached.

In fact, if $f^{-1}[a,b]$ contains K nondegenerate critical points of indices $\lambda_1, \dots, \lambda_K$, M^b has the homotopy type of M^a with K cells of dimensions $\lambda_1, \dots, \lambda_K$ attached.