A little more on Lie Groups, the complex case, and the polar form revisited.

We saw before that every matrix A in $GL(n;\mathbb{R})$ could be written in <u>polar form</u> as

$$A = RS \leftarrow \text{symmetric positive definite.}$$

$$\uparrow$$

orthogonal

Of course, now that we know the matrix exponential, I can't resist telling you:

Theorem. (Gallier, Diff G, 1.8)
The exponential map is a homeomorphism from symmetric matrices to symmetric, positive definite matrices.

so we have an even nicer
Polar Form Theorem. For every real invertible matrix A $\exists$ a unique orthogonal R and ~~symm. pos. def~~ symmetric S so that $A = Re^S$.

We now want to say something about the complex case. As before,

Proposition. The Lie algebra $\mathfrak{u}(n)$ consists of skew Hermitian ($A^* = -A$) matrices, while $\mathfrak{su}(n)$ consists of skew Hermitian matrices with trace zero.

The proof is as before, but remember that the trace of a skew Hermitian matrix is generally purely imaginary, not zero.

Exercise. Prove it for $\mathfrak{su}(n)$.

Note: $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ are not complex vector spaces (as multiplying by i makes a skew-Hermitian matrix into a Hermitian one!)

Theorem. $\exp: \mathfrak{u}(n) \to U(n)$ and $\exp: \mathfrak{su}(n) \to SU(n)$ are surjective.

Similarly, we get

Theorem. The exp map is a homeomorphism from Hermitian matrices to Hermitian pos. definite matrices.

and

Polar Form Theorem. For every invertible complex matrix A, we have a unique unitary U and Hermitian S so that $A = Ue^S$.

Now we can count some ~~complex~~ real dimensions for complex Grassmannians.

1) Dimension of Hermitian matrices

$$= \underbrace{2\left(\frac{n(n-1)}{2}\right)}_{\substack{\text{complex off} \\ \text{diagonal entries}}} + \underbrace{n}_{\substack{\text{real diagonal} \\ \text{entries}}} = n^2$$

2) $GL(n, \mathbb{C}) \cong U(n) \times \mathbb{R}^{n^2}$

3) $\dim U(n) = 2n^2 - n^2 = n^2.$

We have

$$V_k(\mathbb{C}^n) = U(n) / U(n-k)$$

so

$$\dim V_k(\mathbb{C}^n) = n^2 - (n-k)^2$$

$$= n^2 - (n^2 - 2nk + k^2)$$

$$= 2nk - k^2 = k(2n-k).$$

and

$$\dim G_k(\mathbb{C}^n) = U(n) / U(k) \times U(n-k)$$

$$= n^2 - k^2 - (n-k)^2$$

$$= n^2 - k^2 - (n^2 + 2nk + k^2)$$

$$= 2nk - 2k^2 = 2k(n-k).$$

4) Note that this corrects our earlier computation to

$$GL(n;\mathbb{C}) \cong U(n) \times \mathbb{R}^{n^2}$$

(because I miscounted dimension for the complex Cholesky, where the entries on the diagonal must be <u>real</u>, too.)

We now understand how to take geodesics <u>from the identity</u> in $O(n)$ or $U(n)$. Luckily, it's not hard to work from any other point:

Theorem. ~~Left~~ Multiplication by a constant matrix in $O(n), U(n)$, is an isometry of $O(n)$ or $U(n)$.

This means that the exponential map <u>at Q</u> can be defined as follows.

1) The tangent space to $O(n)$ at Q is the space of matrices $Q\Delta$ where Δ is skew-symmetric (Hermitian).

2) The map

$$\exp_Q : T_Q O(n) \to O(n)$$

is defined by

$$Q\Delta \mapsto Qe^{\Delta}.$$

Given this, we now ~~make~~ recall an important ~~observation:~~ theorem:

Theorem. If $f: M \to N$ is a Riemannian submersion and $\gamma(t)$ is a geodesic on M so that $\gamma'(t) \in \mathrm{Horiz}_{\gamma(t)}$ for all t, then $\pi(\gamma(t))$ is a geodesic on N.

We are almost able to compute geodesics on our manifolds!

Lemma. Given a family of matrices $A(t)$ parametrized by t,

$$\frac{d}{dt} e^{A(t)} = A'(t)\, e^{A(t)}$$

Proof. Differentiate term-by-term.

Proposition. The ~~of~~ geodesics on $V_k(\mathbb{C}^n)$ and $V_k(\mathbb{R}^n)$ are all images of $U(n)$ and $O(n)$ geodesics under $\pi: U(n) \to V_k(\mathbb{C}^n)$ and $\pi: O(n) \to V_k(\mathbb{R}^n)$.

Proof. ~~the~~ Suppose we have a matrix $Q \in O(n)$, ~~so that~~ and a tangent vector $Q\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix} \in \mathrm{Horiz}_Q$.

The geodesic in this direction is then

$$Q(t) = Q e^{t\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}}$$

so we can see that

$$Q'(t) = \left(Q e^{t\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}}\right)\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix}$$

$$= Q(t)\begin{bmatrix} A & -B^T \\ B & 0 \end{bmatrix},$$

and in particular, that $Q'(t) \in \mathrm{Horiz}_{Q(t)}$. □

A version of this proof works just as well for the Grassmann manifolds.

Proposition. The geodesics on $G_k(\mathbb{R}^n)$ and $G_k(\mathbb{C}^n)$ are projections of $O(n)$ and $U(n)$ geodesics, and also projections of ~~and~~ $V_k(\mathbb{R}^n)$ and $V_k(\mathbb{C}^n)$ geodesics.

Note: How did we know these are <u>all</u> the geodesics? Well, the horizontal space is isomorphic to the ~~base~~ tangent space of the target manifold in a submersion, so the geodesics in, say, $O(n)$ with horizontal tangents project to the space of $V_k(\mathbb{R}^n)$ geodesics with <u>all</u> tangents.

Since any geodesic is <u>uniquely</u> determined by basepoint and tangent, these are <u>all</u> the geodesics on $V_k(\mathbb{R}^n)$.

Examples, on computer.