Topology of Grassmann and Stiefel Mflds.

We start by reviewing the basic idea of fiber bundles. Much like covering spaces, fiber bundles are defined by a lifting property:

Definition. $p: E \to B$ has the <u>homotopy lifting property</u> with respect to X if any homotopy $g_t: I \times X \to B$ ~~so that~~ with a $\tilde{g}_0 : X \to E$ ~~has an extension~~ ~~lifts~~ that makes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{g}_0"] \arrow[r, "g_0"'] & B \end{tikzcd}$$

commute has an extension $\tilde{g}_t : X \times I \to E$ which makes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \times I \arrow[ur, "\tilde{g}_t"] \arrow[r, "g_t"'] & B \end{tikzcd}$$

commute. Such a $\tilde{g}_t$ is said to be a <u>lift</u> of g_t (with respect to p).

If $p: E \to B$ has the homotopy lifting property with respect to all ~~base~~ spaces X, we say it is a fibration.

We care (largely) because of

Homotopy exact sequence of a fibration.
Suppose $p: E \to B$ is a fibration, B is path-connected, and we have a basepoint $b_0 \in B$ and a basepoint x_0 in $p^{-1}(b_0) = F$.
Then there is a long exact sequence

$$\to \pi_n(F, x_0) \to \pi_n(E, x_0) \to \pi_n(B, b_0) \to$$

$$\to \pi_{n-1}(F, x_0) \to \pi_{n-1}(E, x_0) \to \cdots\cdots$$

$$\cdots\cdots \to \pi_0(E, x_0) \to 0$$

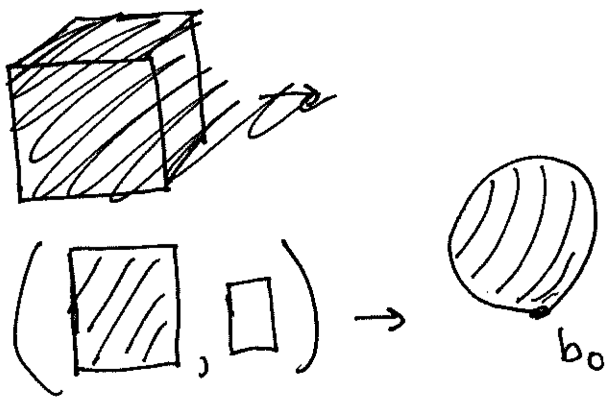
The interesting map here is clearly

$$\to \pi_n(B, b_0) \longrightarrow \pi_{n-1}(F, x_0) \to$$

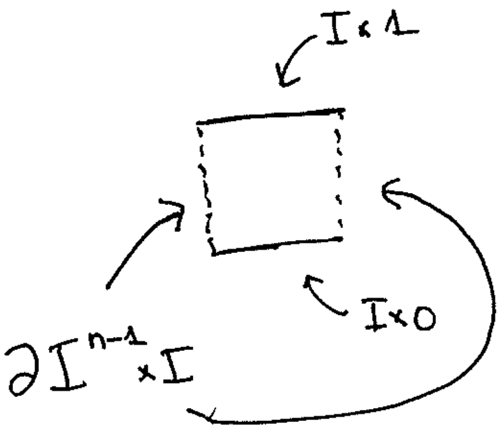
called the <u>connecting homomorphism</u>.

Consider an element in $\pi_n(B, b_0)$. This is a map $S^n \to B$ or, alternately, a map

$$f: (I^n, \partial I^n) \to (B, b_0)$$



Now we can write $I^n = I^{n-1} \times I$, and $\partial I_n = I^{n-1} \times \{0,1\} \cup \partial I^{n-1} \times I$. In pictures,

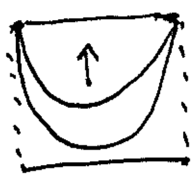


Now map the subset $\partial I^{n-1} \times I \cup I \times \{0\}$ to $x_0 \in F$. This is a lift of part of f.

so it extends to a lift

$\tilde{f} : I^{n-1} \times I \longrightarrow E$

We can picture this as



Now the restriction $\tilde{f}\big|_{I^{n-1} \times \{1\}}$ has to be contained in $F = p^{-1}(b_0)$ and have $\tilde{f}\big|_{\partial I^{n-1} \times \{1\}} = x_0$ (both by lifting property).

But then this is a map

$$(I^{n-1}, \partial I^{n-1}) \to (F, x_0)$$

and hence an element of $\pi_{n-1}(F, x_0)$, as desired. (see Hutchings notes, on page 20).

We usually use this idea in a special case:

Definition. A <u>fiber bundle</u> ~~is a~~ structure on E with base B and fiber F is a projection map $p: E \to B$ so that each $b \in B$ has a neighborhood $\mathcal{U}$ so that there is a homeomorphism $h: p^{-1}(\mathcal{U}) \to \mathcal{U} \times F$ so

$$\begin{tikzcd} E \supset p^{-1}(\mathcal{U}) \arrow[rr, "h"] \arrow[dr, "p"'] & & \mathcal{U} \times F \arrow[dl, "\pi"] \\ & B \supset \mathcal{U} & \end{tikzcd}$$

This map is called a <u>local trivialization</u> of the bundle structure.

Examples. If F is discrete, a fiber bundle is a covering space.

If we take the Hopf map: $U\mathbb{H} \to UI\mathbb{H}$ defined by $q \mapsto \bar{q} i q$, this the projection of a bundle $S^1 \to S^3 \xrightarrow{p} S^2$.

$$F \to E \longrightarrow B$$

Proposition. (see Hatcher, p 379)
Every fiber bundle over a paracompact base space is a fibration. (For instance, every fiber bundle over a CW complex is a fibration.)

We can now use the ~~long~~ homotopy long exact sequence of a fibration to compute homotopy groups.

Example. The Hopf bundle $S^1 \to S^3 \to S^2$.
We know that $\pi_k(S^1) = \begin{cases} \mathbb{Z}, & k=1 \\ 0, & \text{otherwise,} \end{cases}$

so this means

$$\to \pi_k(S^1) \to \pi_k(S^3) \to \pi_k(S^2) \to \pi_{k-1}(S^1) \to$$

shows $\pi_k(S^3) \cong \pi_k(S^2)$ for $k>2$.
In particular, $\pi_3(S^3) = \mathbb{Z} = \pi_3(S^2)$. The rest of the sequence looks like

$$\to \underset{\substack{\| \\ 0}}{\pi_2(S^1)} \to \underset{\substack{\| \\ 0}}{\pi_2(S^3)} \to \underset{\substack{\| \\ \mathbb{Z}}}{\pi_2(S^2)} \to \underset{\substack{\| \\ \mathbb{Z}}}{\pi_1(S^1)} \to \underset{\substack{\| \\ 0}}{\pi_1(S^3)} \to \cdots$$

and so (apart from the fact that the generator in $\pi_1(S^1)$ is the image under the connecting homomorphism of the generator in $\pi_2(S^2)$) doesn't say much.

We now apply these ideas to our area of interest. We have fiber bundles

$$O(n) \to V_n(\mathbb{R}^K) \to G_n(\mathbb{R}^K)$$
$$U(n) \to V_n(\mathbb{C}^K) \to G_n(\mathbb{C}^K)$$
$$Sp(n) \to V_n(\mathbb{H}^K) \to G_n(\mathbb{H}^K)$$

and for that matter, similar bundles when $K = \infty$.

Exercise. Prove that these are fiber bundles.

Proposition. $V_n(\mathbb{R}^\infty)$, $V_n(\mathbb{C}^\infty)$, $V_n(\mathbb{H}^\infty)$ are all contractible.

Proof. First, define a homotopy $h_t : \mathbb{R}^\infty \to \mathbb{R}^\infty$ by $(x_1, x_2, x_3, \ldots) \mapsto (1-t)(x_1, \ldots) + t(0, x_1, \ldots)$. This is a linear map with no kernel for each t, so it is a homotopy from any n-frame to another n ~~frame~~ linearly independent vectors with first coordinate 0.

This can be orthonormalized continuously by Gram-Schmidt, and so

$$V_n(\mathbb{R}^\infty) \xrightarrow[\text{retracts onto}]{} \text{n-frames with first coordinate zero}$$

$$\downarrow \text{repeat}$$

n-frames with first n coordinates 0

Now we can deform n-frames $v_1, \dots, v_n$ with first n-coordinates zero into $e_1, \dots, e_n$ by continoously orthogonalizing

$$(1-t)(v_1, \dots, v_n) + t(e_1, \dots, e_n),$$

which retracts $V_n(\mathbb{R}^\infty)$ onto $\{(e_1, \dots, e_n)\}$. $\square$.

Thus

$$\pi_i(O(n)) = \pi_{i+1}(G_n(\mathbb{R}^\infty))$$

$$\pi_i(U(n)) = \pi_{i+1}(G_n(\mathbb{C}^\infty))$$

by the long exact sequence. For our favorite $G_2(\mathbb{C}^\infty)$ (~~sp~~ framed space curves mod simultaneous rotation of the frame),

$$\pi_1(U(2)) = \pi_2(G_2(\mathbb{C}^\infty)) = \mathbb{Z}$$

$$\pi_K(U(2)) = \pi_K(S^3) \quad \text{for } K \geq 2,$$

so

$$\pi_2(U(2)) = \pi_3(G_2(\mathbb{C}^\infty)) = 0.$$

$$\pi_3(U(2)) = \pi_4(G_2(\mathbb{C}^\infty)) = \mathbb{Z}$$

$$\pi_4(U(2)) = \pi_5(G_2(\mathbb{C}^\infty)) = \mathbb{Z}/2\mathbb{Z},$$

and so forth. There is a lot of tasty topology here!

For plane curves, $G_2(\mathbb{R}^\infty)$, the situation is a lot more boring, with the only nontrivial homotopy group

$$\pi_1(O(2)) = \mathbb{Z} = \pi_2(G_2(\mathbb{R}^\infty)).$$

Research problem: Explain the generator of ~~plane~~ π_2 of plane curves and the generator of π_2 $\cancel{G_2}$ of space curves. (Conjecture: The second one is linking number of the pushoff with the base.)