Geometric Inequalities.

Now that we have a language to describe curves we can start to put numbers on our intuitions about curve geometry.

Proposition. [Buck-Simon]

If γ is a curve of length L contained in a ball of radius R, then

$$\int \kappa(s)\, ds \geq \frac{L}{R} - 2.$$

Proof. Wlog, let the ball be centered at $\vec{0}$. Now

$$L = \int_0^L \langle \gamma'(s), \gamma'(s) \rangle \, ds$$

$$= \langle \gamma'(s), \gamma(s) \rangle \Big|_{s=0}^{L} - \int_0^L \langle \gamma^{\cancel{\prime}}(s), \gamma''(s) \rangle \, ds$$

(from integration by parts)

Now $|\gamma(s)| \leq R$ and $|\gamma'(s)| = 1$, so

$$\left| \langle \gamma'(s), \gamma(s) \rangle \right| \leq R$$

and

$$\langle \gamma'(s), \gamma(s) \rangle \Big|_{s=0}^{L} < 2R.$$

Further, $|\gamma'(s)| = 1$, so $\left| \langle \gamma'(s), \gamma''(s) \rangle \right| \leq \overset{RK(s).}{\overset{\shortparallel}{|\gamma''(s)|R}}$

This means that

$$\left| \int_0^L \langle \gamma'(s), \gamma''(s) \rangle \, ds \right| \overset{(*)}{\leq} \int_0^L \left| \langle \gamma'(s), \gamma''(s) \rangle \right| ds$$

$$\leq \int_0^L RK(s) \, ds = R \int_0^L K(s) \, ds.$$

The middle step with the $(*)$ is a special case of Jensen's inequality, which states

$$g\left(\int f(x) \, dx \right) \leq \int g(f(x)) \, dx$$

for any concave up function g.

Assembling the pieces,

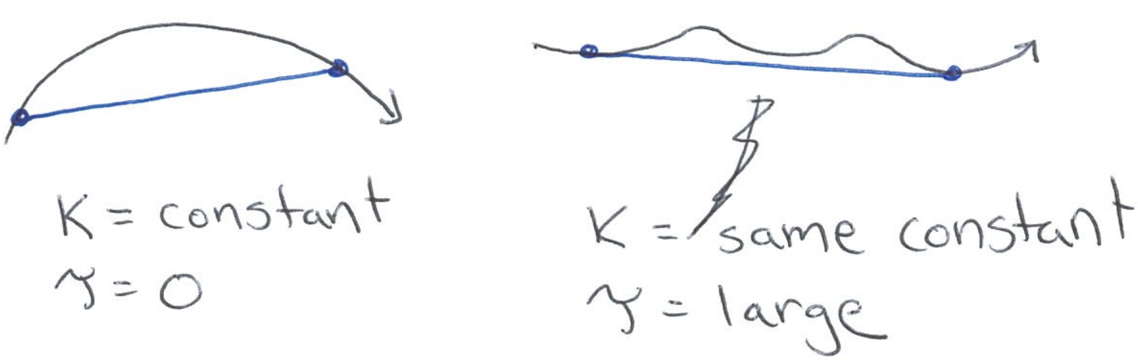
$$L \leq 2R + R \int \kappa(s)\, ds$$

and we need only solve for $\int \kappa(s)\, ds$. □

We note that there's a stronger result

$$\int \kappa(s)\, ds \geq \frac{L}{R}$$

called "Chakerian's inequality" (1962). (But the proof is harder.) We now consider a different intuition:



If a space curve and a plane curve have the same curvature (pointwise), the plane curve bends "more effectively".

The theorem version is better:

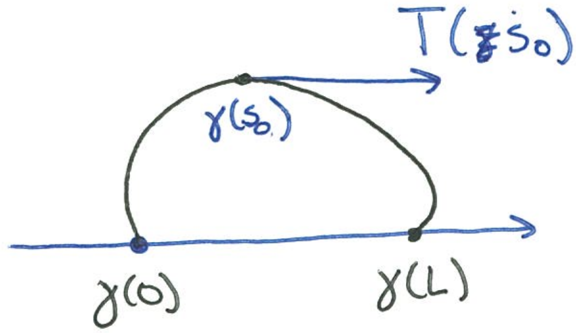
Theorem [Schur's Theorem]

If γ is a plane curve of length L with curvature $\kappa(s)$ and γ^* is a space curve with curvature $\kappa^*(s) \leq \kappa(s)$, and ~~forms~~ the closed curve given by joining $\gamma(L)$ to $\gamma(0)$ with a line segment is convex, then

$$|\gamma(L) - \gamma(0)| \leq |\gamma^*(L) - \gamma^*(0)|,$$

with equality $\Leftrightarrow$ the curves are congruent.

Proof. Since



the arc of γ and the chord $\gamma(0)\gamma(L)$ are convex, $\exists$ some s_0 so $T(s_0)$ is parallel to the chord. ~~T(s₀) is~~

Consider the functions

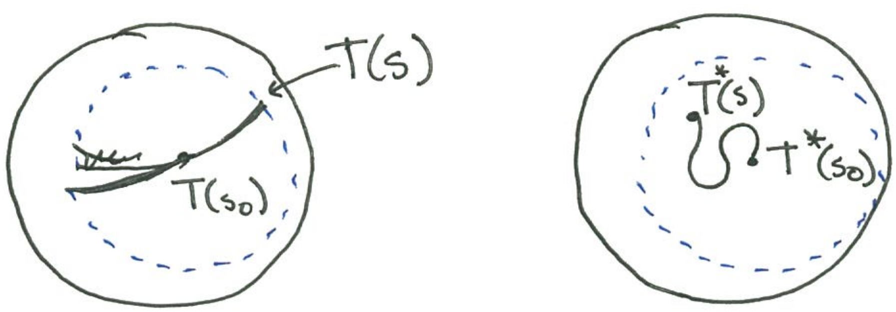
$$f(s) = \langle \gamma(s), T(s_0) \rangle \qquad f^*(s) = \langle \gamma^*(s), T^*(s_0) \rangle$$

Wlog, we may assume $\gamma(0) = \gamma^*(0) = 0$. We claim that

$$f'(s) = \langle T(s), T(s_0) \rangle \qquad (f^*)'(s) = \langle T^*(s), T^*(s_0) \rangle$$

$$= \cos \angle T(s), T(s_0) \leq \qquad = \cos \angle T^*(s), T^*(s_0).$$

To see this, consider the picture on S^2,



The angle $\angle T(s_0), T(s)$ = spherical distance from $T(s_0)$ to $T(s)$ = arclength of T between s_0 and s because γ is convex and planar, so T follows a great circle arc.

On the other hand, this arclength is

$$\int_{s_0}^{s} |T'(s)|\,ds = \int_{s_0}^{s} \kappa(s)\,ds \geq \int_{s_0}^{s} \kappa^*(s)\,ds = \int_{s_0}^{s} |T^{*\prime}(s)|\,ds$$

and hence greater than the arclength of T^* between s_0 and s (which may not follow a great circle in addition!). So

$$\angle T^*(s), T^*(s_0) \leq \angle T(s), T(s_0)$$

(at least, as long as the angles are $\leq \pi$, which they are... do you see why?). Since $\cos$ is decreasing on $[0, \pi]$, this shows

$$f'(s) = \cos \angle T(s), T(s_0) \leq \cos \angle T^*(s), T^*(s_0) = f'^*(s)$$

as desired. Now this implies

$$f(L) \leq f^*(L)$$

But

$$f(L) = \langle \gamma(L), T(s_0) \rangle$$
$$= |\gamma(L)|, \text{ since } T(s_0) \text{ is parallel to } \gamma(L).$$

and

$$f^*(L) = \langle \gamma^*(L), T^*(s_0) \rangle$$
$$\leq |\gamma^*(L)|$$

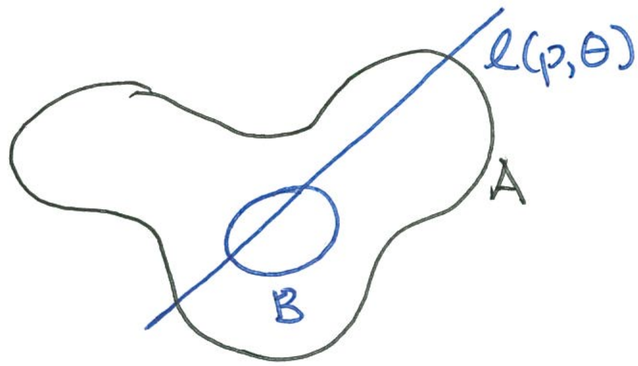
so

$$|\gamma(L)| \leq |\gamma^*(L)|, \text{ as desired. } \blacksquare$$

~~This theorem has lots of interesting consequences. Here are two:~~

We now give a geometric inequality from our line formula.

Proposition. Suppose A is a closed plane curve and $B \subset A$ is a convex plane curve. $\text{Length}(A) \geq \text{Length}(B)$, with $=$ $\iff$ $A = B$.



Proof. Any line through B cuts B twice, and A at least twice. Thus

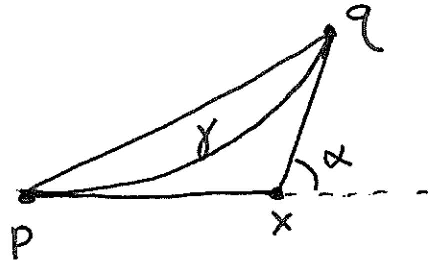
$$\text{Length}(B) = 4 \int_{\ell(p,\theta)} \#\text{ intersections of } \ell(p,\theta), B \; dp\, d\theta$$

$$= 4 \int_{\ell(p,\theta) \text{ cutting } B} 2 \, dp\, d\theta \leq 4 \int_{\substack{\ell(p,\theta) \\ \text{cutting } A}} 2 \, dp\, d\theta$$

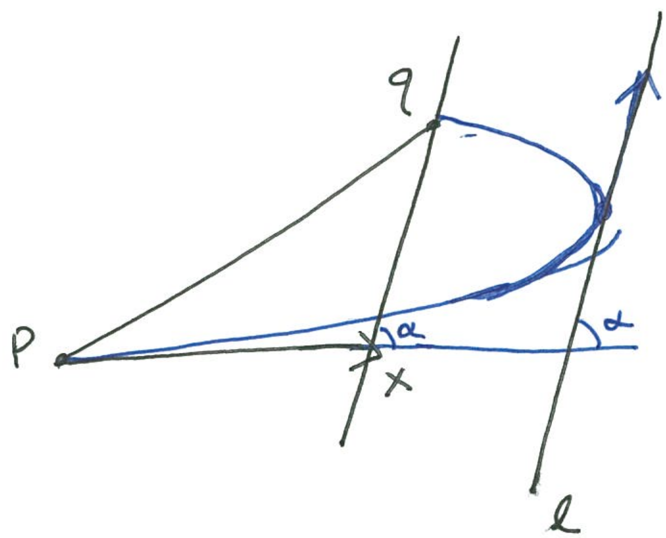
$$\leq 4 \int_{\ell(p,\theta)} \#\text{ intersections of } \ell(p,\theta), A \; dp\, d\theta = \text{Length}(A).$$

Here's another calculus-based inequality using the previous one.

Proposition. Suppose γ is a convex arc of a plane curve with total curvature α, passing through p and q. There is a unique x along the tangent line to γ at p so pxq has turning angle α. γ is contained in Δpxq and has length less than arc pxq.



Proof. Consider the lines parallel to xq. If γ leaves the triangle, it must be tangent to one such line ℓ.



But the spherical distance from $T(p)$ ~~on~~ to the tangent to ~~the~~ ℓ is already α, so γ must have total curvature $> \alpha$ to bend more to come back to xq.

Note γ cannot leave through px (it would have to recross the line px, meaning it touched $px > 2$ times, to reach q. This violates convexity.) or pq (touching this line at q again violates convexity).

Now γ and $\overline{pq}$ are a convex curve inside Δpxq, so γ and $\overline{pq}$ are shorter.

Subtracting $\overline{pq}$, we see

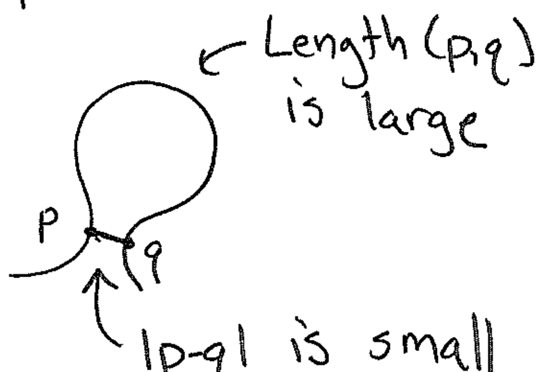
$$\text{Length } \gamma \leq \text{Length } pxq. \qquad \square$$

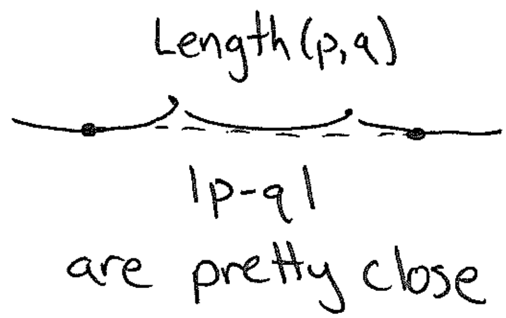
We now introduce

Definition. The distortion of ~~an arc~~ curve γ is given by

$$\delta(\gamma) = \underset{p,q \text{ on } \gamma}{\text{supremum}} \frac{\text{Length}(p,q)}{|p-q|}$$

Examples.





$\delta(\gamma)$ is large

$\delta(\gamma)$ close to 1.

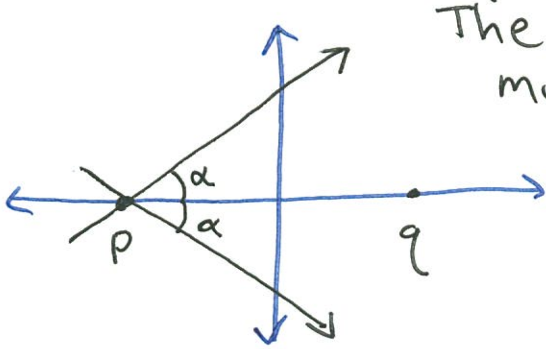
Proposition. [Sullivan, 2007]

Any γ of total curvature $\alpha < \pi$ has distortion at most $\sec(\alpha/2)$.

Proof. Wlog, we can assume the max distortion of all p, q on the arc is at endpoints (or shorten the arc!).

The curve γ has some curvature $\kappa(s)$, by Schur's lemma, replacing γ by the planar convex curve with the same pointwise curvature makes $|p-q|$ larger (and δ smaller). So wlog, γ is convex and planar.

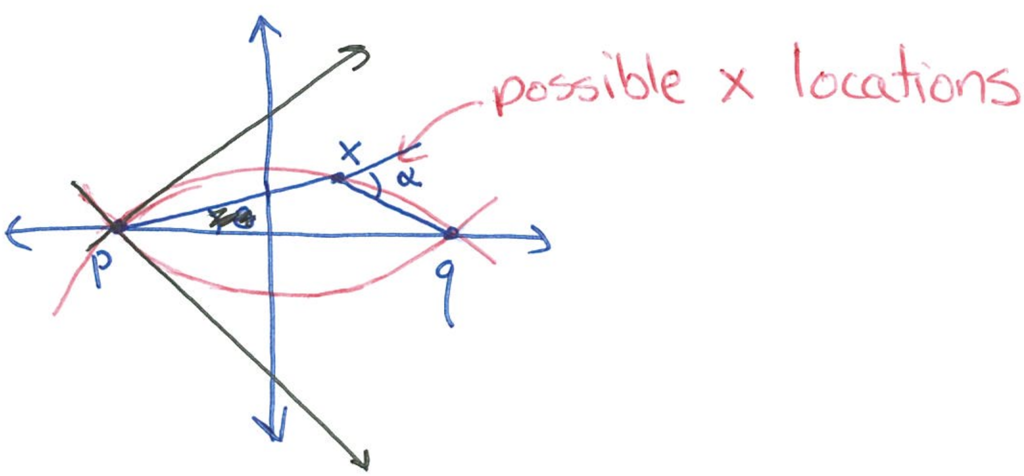
Now fix p and q to be $(-1,0)$ and $(1,0)$.



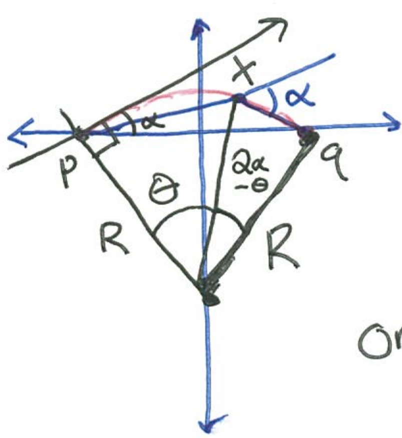
The tangent to γ at p must lie in the sector shown. ~~For each choice, there is a unique x~~

or the arc γ cannot turn enough to get back to q.

Each choice of tangent defines a triangle point x, and these x lie along the circle through p and q with tangent shown.



For $x(\theta)$, the length of pxq is given by using some ~~geo~~ trig: The total angle turned by the tangent to circle is 2α. If we go only θ of that distance,



The length of px and xq are lengths of chords of angle θ, $2\alpha-\theta$. The chord length formula is $2R\sin\frac{\varphi}{2}$ so the length is

$$2R\sin\left(\frac{\theta}{2}\right)+2R\sin\left(\alpha-\frac{\theta}{2}\right)=f(\theta).$$

~~Using angle identities, this is~~

$$\cancel{R\sin(\theta/2)+R\sin\alpha\cos(\theta/2)-R\sin(\theta/2)\cos\alpha}$$

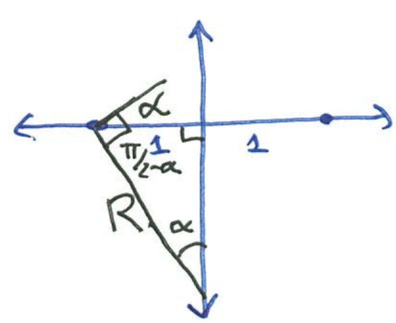
So

$$f'(\theta)=\frac{2R}{2}\cos\frac{\theta}{2}-\frac{2R}{2}\cos(\alpha-\theta/2)$$

and the only critical point is $\theta=\alpha/2$. At this point our length bound is

$$4R\sin\frac{\alpha}{2}=f\left(\frac{\alpha}{2}\right).$$

Now the length $R=\frac{1}{\sin\alpha}=\csc\alpha$.



or

$$R = \frac{1}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}$$

So the length bound

$$f\left(\frac{\alpha}{2}\right) = \frac{4\cancel{\alpha}\,\cancel{\sin\alpha/2}}{\cancel{2}\,\cancel{\sin\alpha/2}\cos\alpha/2} = 2\sec(\alpha/2).$$

Since we have scaled the distance ~~from p to~~ $|p-q|=2$, this makes

$$\delta(\gamma) \leq \frac{2\sec(\alpha/2)}{2} = \sec(\alpha/2). \quad \square$$