Geodesics as Straightest Paths

We start with a lemma:

Lemma. If X and Y are parallel vector fields on $\alpha(t)$ then $\|X\|$, $\|Y\|$ and the angle between X and Y are constant in t.

Proof. Consider

$$\frac{d}{dt} \langle X(\alpha(t)), Y(\alpha(t)) \rangle = \langle X', Y \rangle + \langle X, Y' \rangle$$

Now $\nabla_{\alpha'(t)} X(\alpha(t)) = \frac{d}{dt} X(\alpha(t)) - \langle \frac{d}{dt} X(\alpha(t)), \vec{n} \rangle \vec{n}$, but since $Y \in T_p M$, this means

$$\langle \tfrac{d}{dt} X, Y \rangle = \langle \nabla_{\alpha'(t)} X, Y \rangle$$

Thus our original

$$\frac{d}{dt} \langle X, Y \rangle = \langle \underbrace{\nabla_{\alpha'(t)} X}_{0,\text{ since } X \text{ parallel}}, Y \rangle + \langle X, \underbrace{\nabla_{\alpha'(t)} Y}_{0,\text{ since } Y \text{ parallel}} \rangle$$

$$= 0 + 0 = 0. \qquad \square$$

We can now define

Definition. A curve $\alpha(t) \in M$ is a geodesic if $\alpha'(t)$ is parallel: $\nabla_{\alpha'(t)} \alpha'(t) = 0$.

By the lemma, this means $|\alpha'(t)|$ is constant, so $\alpha(t)$ is parametrized by a constant multiple of arclength.

We can write ~~$\alpha(t)$~~ κN in the Darboux frame $\{T, \vec{n} \times T, \vec{n}\}$ (where $\vec{n}$ is the surface normal) as

$$\kappa N = \underbrace{\langle \kappa N, \vec{n} \times T \rangle}_{\text{geodesic curvature } \kappa_g} \vec{n} \times T + \underbrace{\langle \kappa N, \vec{n} \rangle}_{\text{normal curvature } \kappa_{\vec{n}}} \vec{n}$$

Lemma. $\alpha(t)$ is a geodesic $\Leftrightarrow \kappa_g = 0$.

We see $\alpha''(t) = \kappa N$, so

$$\nabla_{\alpha'(t)} \alpha'(t) = \kappa N - \langle \kappa N, \vec{n} \rangle \vec{n}.$$

This is 0 exactly when κ_g vanishes.

We can now simplify our earlier equations for parallel transport by observing that $a(t) = u'(t)$, $b(t) = v'(t)$. We get

$$u''(t) + \Gamma^u_{uu}\, u'(t)^2 + 2\Gamma^u_{uv}\, u'(t)v'(t) + \Gamma^u_{vv}\, v'(t)^2 = 0$$

$$v''(t) + \Gamma^v_{uu}\, u'(t)^2 + 2\Gamma^v_{uv}\, u'(t)v'(t) + \Gamma^v_{vv}\, v'(t)^2 = 0.$$

Proposition. Given $p \in M$ and $\vec{v} \in T_p M \neq \vec{0}$, there is an $(-\epsilon, \epsilon)$ and a unique geodesic $\alpha(t)$ with $\alpha(0) = P$ and $\alpha'(0) = V$.

This is again a consequence of our theorem on solutions of ODEs.

~~Idea~~

Example. Let ~~x be the~~ $X(u,v) = (u, v, 0)$ parametrize the plane. Since $X_{uu} = X_{vv} = 0$, all Christoffel symbols are O and the geodesic equations are $u'' = 0$, $v'' = 0$.

Example. Revisiting the sphere

$$X(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

we recall that all Christoffel symbols were zero except $\Gamma_{uv}^{v} = \cot u$, $\Gamma_{vv}^{u} = -\sin u \cos u$.
So we get

$$u''(t) - \sin u(t)\cos u(t)\, v'(t)^2 = 0$$

$$v''(t) + 2\cot u(t)\, u'(t)\, v'(t) = 0.$$

One set of solutions is $v(t) = v_0$, $u(t) = t$. These give great circles through the north pole!

We'll have to work for everything else. Now separating variables,

$$\frac{v''(t)}{v'(t)} = -2\cot u(t)\, u'(t) = \frac{-2(\cos u)\, u'}{\sin u}$$

so (integrating)

$$\ln v'(t) = -2\ln \sin u + C$$

$$= -2 \ln(C \sin u) \qquad \text{(c changed!)}$$

$$= \ln\left(\frac{C}{\sin^2 u}\right) \qquad \text{(c changed again!)}$$

So

$$V'(t) = \frac{C}{\sin^2 u} = C \csc^2 u$$

Now if we plug this into the first equation,

$$u'' \cancel{} - \frac{\sin u \cos u \, C^2}{\sin^{\cancel{4}3} u} = 0$$

So

$$u'' = \frac{C^2 \cos u}{\sin^3 u}$$

Here's a neat trick. We'd like to integrate both sides, ~~but don't~~ wrt u, but don't have a $u'(t)$ on either. Why not add one? Multiplying through by $u'(t)$,

we get

$$u'' \cdot u' = \frac{c^2 \cos u}{\sin^3 u} u'$$

or (integrating w.r.t u)

$$\frac{(u')^2}{2} = \frac{c^2}{-2} \sin^{-2} u + C$$

Simplifying, we get

$$(u')^2 = C - \frac{c^2}{\sin^2 u} \qquad (C \text{ changed})$$

or

$$u' = \pm \sqrt{C - \frac{c^2}{\sin^2 u}}$$

We now write

$$\frac{dv}{du} = \frac{v'(t)}{u'(t)} = \pm \frac{c \csc^2 u}{\sqrt{c^2 - c^2 \csc^2 u}}$$

Now this is more hopeful than it looks!

$$\sin^2 u + \cos^2 u = 1, \text{ so } 1 + \cot^2 u = \csc^2 u$$

Also,

$$\frac{d}{du} \cot u = -\csc^2 u.$$

So first, we have

$$dv = \pm \frac{c \csc^2 u \, du}{\sqrt{C^2 - c^2(1 + \cot^2 u)}}$$

$$= \pm \frac{c \csc^2 u \, du}{\sqrt{(C^2 - c^2) - c^2 \cot^2 u}}$$

Now let's make a clever change of variables:

$$c \cot u = \sqrt{C^2 - c^2} \sin w$$

$$-c \csc^2 u \, du = \sqrt{C^2 - c^2} \cos w \, dw$$

Now

$$c^2 \cot^2 u = (C^2 - c^2) \sin^2 w,$$

so

$$\sqrt{(C^2 - c^2) - c^2 \cot^2 u} = \sqrt{(C^2 - c^2) - (C^2 - c^2) \sin^2 w}$$

$$= \sqrt{C^2 - c^2} \sqrt{1 - \sin^2 w} = \sqrt{C^2 - c^2} \cos w.$$

Thus

$$dv = \pm \frac{\csc^2 u \, du}{\sqrt{(C^2-c^2) - c^2 \cot^2 u}}$$

$$= \pm \mp \frac{\sqrt{C^2-c^2} \cos \omega \, d\omega}{\sqrt{C^2-c^2} \cos \omega} = \mp d\omega.$$

and (amazingly) we have

$$\omega = \pm V + a$$

for some constant a. Thus

$$c \cot u = \sqrt{C^2-c^2} \sin \omega$$

$$= \sqrt{C^2-c^2} \sin (\pm V + a)$$

$$= \sqrt{C^2-c^2} (\sin a \cos V \pm \cos a \sin V)$$

or

$$c \cos u = \sqrt{C^2-c^2} \sin u (A \cos V + B \sin V).$$

Now you've undoubtedly forgotten what
u and v mean at this point, but

recall that

$$X(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

so we can call these X, Y, Z and observe the equation is

$$cZ = (A\sqrt{c^2-c^2})X + (B\sqrt{c^2-c^2})Y,$$

or the equation implies α lies on a plane through the origin (with <u>constant</u> equation) and hence $\alpha(t)$ is a great circle.

Conclusion. There are no other ~~great~~ ~~circle~~ geodesics on S^2!

Of course, you could probably have gotten that from uniqueness, but this was more fun. ☺