More on framings, curvature and torsion.

We have now derived the Frenet frame and its derivative functions curvature and torsion, for an arclength parametrized curve.

Our goal is to extend this to an arbitrary parametrization.

We first recall an idea from linear algebra: Gram-Schmidt orthogonalization.

Algorithm. Given linearly independent vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n \in \mathbb{R}^n$, we can construct an orthonormal basis $\vec{\omega}_1, \ldots, \vec{\omega}_n$ for $\mathbb{R}^n$ recursively as follows.

1) Let $\vec{\omega}_1 = \dfrac{\vec{v}_1}{\|\vec{v}_1\|}$.

2) Having constructed $\vec{\omega}_1, \ldots, \vec{\omega}_k$, let

$$\vec{u}_{k+1} = \vec{v}_{k+1} - \langle \vec{v}_{k+1}, \vec{\omega}_k \rangle \omega_k - \cdots - \langle \vec{v}_{k+1}, \vec{\omega}_1 \rangle \vec{\omega}_1$$

$$= \vec{v}_{k+1} - \mathrm{proj}_{\mathrm{span}(\omega_1, \ldots, \omega_k)} \vec{v}_{k+1}$$

and let

$$\vec{\omega}_{k+1} = \frac{\vec{u}_{k+1}}{\|\vec{u}_{k+1}\|}.$$

Theorem. Let $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$ be a parametrized curve. At each t, let

$$\vec{F}_1(t), \ldots, \vec{F}_{n-1}(t)$$

be the result of Gram-Schmidt orthogonalization on

$$\vec{\alpha}'(t), \ldots, \vec{\alpha}^{(n-1)}(t).$$

The <u>Frenet frame</u> at t is the frame $\vec{F}_1(t), \ldots, \vec{F}_{n-1}(t), \vec{F}_n(t)$ where $\vec{F}_n(t)$ is the unique unit vector so

$$\begin{pmatrix} \uparrow & & \uparrow \\ F_1 & \cdots & F_n \\ \downarrow & & \downarrow \end{pmatrix}$$ is an orthogonal matrix with determinant $+1$.

This frame exists $\Leftrightarrow \vec{\alpha}'(t), \ldots, \vec{\alpha}^{(n-1)}(t)$ are linearly independent.

Example. $\vec{v}_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\vec{u}_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \left\langle \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \right\rangle \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \frac{1}{10} \cdot 8 \cdot \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 - 24/10 \\ 2 - 8/10 \end{pmatrix} = \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}$$

We can then compute

$$\vec{w}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|} = \frac{1}{\sqrt{\frac{4}{25} + \frac{36}{25}}} \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}$$

$$= \frac{1}{\frac{2\sqrt{10}}{5}} \begin{pmatrix} -2/5 \\ 6/5 \end{pmatrix}$$

$$= \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

We now use this theorem to find the Frenet frame for a curve $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^3$.

Observation. $T(t)$ and $N(t)$ are in the plane spanned by $\vec{\alpha}'(t)$ and $\vec{\alpha}''(t)$. Therefore

$$B(t) = \pm \frac{\vec{\alpha}'(t) \times \vec{\alpha}''(t)}{\|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|}$$

Observation. $T(t) = \dfrac{\vec{\alpha}'(t)}{\|\vec{\alpha}'(t)\|}$.

The hardest case is $N(t)$. Recalling $\begin{smallmatrix} \nearrow & T & \searrow \\ B & \leftarrow & N \end{smallmatrix}$, we compute

$$N(t) = B(t) \times T(t) = \pm \frac{\cancel{\pm}(\vec{\alpha}'(t) \times \vec{\alpha}''(t)) \times \vec{\alpha}'(t)}{\|\vec{\alpha}'(t)\| \, \|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|}$$

We now consider

$$(\vec{\alpha}' \times \vec{\alpha}'') \times \vec{\alpha}' = -\vec{\alpha}' \times (\vec{\alpha}' \times \vec{\alpha}'')$$

$$= -\vec{\alpha}' \langle \vec{\alpha}', \vec{\alpha}'' \rangle + \vec{\alpha}'' \langle \vec{\alpha}', \vec{\alpha}' \rangle$$

allowing us to compute

$$N(t) = \mp \left(\frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle}{\| \vec{\alpha}' \times \vec{\alpha}'' \|} \frac{\vec{\alpha}'}{\| \vec{\alpha}' \|} - \frac{\| \vec{\alpha}' \|}{\| \vec{\alpha}' \times \vec{\alpha}'' \|} \vec{\alpha}'' \right)$$

We can now compute

$$\kappa(t) = \left\langle \frac{d}{ds} T(t), N(t) \right\rangle$$

$$\tau(t) = -\left\langle \frac{d}{ds} B(t), N(t) \right\rangle$$

To do this, we have to be able to compute

$$\frac{d}{ds} T(t) = T'(t) \cdot \frac{dt}{ds}$$

and

$$\frac{d}{ds} B(t) = B'(t) \cdot \frac{dt}{ds}.$$

We can compute $\frac{dt}{ds}$ bey remembering that $t(s)$ is the inverse function of $s(t)$, and that

$$s(t) = \int_0^t \|\vec{\alpha}'(x)\| \, dx$$

so

$$s'(t) = \|\vec{\alpha}'(t)\|$$

and hence

$$t'(s) = \frac{1}{\|\vec{\alpha}'(t)\|}.$$

Now we just compute.

$$\frac{d}{dt}T = \frac{d}{dt}\frac{\vec{\alpha}'}{\|\vec{\alpha}'\|}$$

$$= \frac{\vec{\alpha}''}{\|\vec{\alpha}'\|} - \frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle}{\|\vec{\alpha}'\|^3}\vec{\alpha}'$$

so

$$\frac{d}{ds}T = \frac{\vec{\alpha}''}{\|\vec{\alpha}'\|^2} - \frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle}{\|\vec{\alpha}'\|^4}\vec{\alpha}'$$

and

$$\left\langle \frac{d}{ds}T, N \right\rangle = \mp\left(\frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle^2}{\|\vec{\alpha}' \times \vec{\alpha}''\| \, \|\vec{\alpha}'\|^3} - \frac{\langle \vec{\alpha}'', \vec{\alpha}'' \rangle}{\|\vec{\alpha}'\| \, \|\vec{\alpha}' \times \vec{\alpha}''\|} \right.$$

$$\left. - \frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle^2 \langle \vec{\alpha}', \vec{\alpha}' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\| \, \|\vec{\alpha}'\|^5} + \frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle^2}{\|\vec{\alpha}' \times \vec{\alpha}''\| \, \|\vec{\alpha}'\|^3} \right)$$

$$= \mp\left(\frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle^2}{\|\vec{\alpha}' \times \vec{\alpha}''\| \, \|\vec{\alpha}'\|^3} - \frac{\langle \vec{\alpha}'', \vec{\alpha}'' \rangle \langle \vec{\alpha}', \vec{\alpha}' \rangle}{\|\vec{\alpha}'\|^3 \, \|\vec{\alpha}' \times \vec{\alpha}''\|} \right)$$

$$= \mp \frac{1}{\|\vec{\alpha}' \times \vec{\alpha}''\| \, \|\vec{\alpha}'\|^3} \left(\langle \vec{\alpha}', \vec{\alpha}'' \rangle^2 - \langle \vec{\alpha}', \vec{\alpha}' \rangle \langle \vec{\alpha}'', \vec{\alpha}'' \rangle \right)$$

Now the quantity in parens looks suspicious; in fact it's related to

$$\|\vec{a} \times \vec{b}\|^2 + \langle \vec{a}, \vec{b} \rangle^2 = \|\vec{a}\|^2 \|\vec{b}\|^2.$$

$$= \pm \frac{\|\vec{\alpha}' \times \vec{\alpha}''\|}{\|\vec{\alpha}'\|^3}$$

which is our formula for curvature in terms of t. Since $\kappa(t) \geq 0$, this (finally!) resolves the sign question, revealing

$$B(t) = \frac{\vec{\alpha}' \times \vec{\alpha}''}{\|\vec{\alpha}' \times \vec{\alpha}''\|}, \quad N(t) = \frac{1}{\|\vec{\alpha}' \times \vec{\alpha}''\|} \left(\|\vec{\alpha}'\| \vec{\alpha}'' - \frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle \vec{\alpha}'}{\|\vec{\alpha}'\|} \right)$$

Now we need only differentiate

$$\frac{d}{dt} \frac{\vec{\alpha}' \times \vec{\alpha}''}{\|\vec{\alpha}' \times \vec{\alpha}''\|}$$

and dot with $N(t)$ to find torsion.

This is a homework problem, but the eventual answer is

$$\tau(t) = \frac{\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle}{\| \vec{\alpha}' \times \vec{\alpha}'' \|^2}$$

We can now do some examples.

Example. Find T, N, B and κ and τ for the curve $\vec{\alpha}(t) = (t, t^2, t^3)$.

We start by computing

$$\vec{\alpha}'(t) = (1, 2t, 3t^2)$$

$$\vec{\alpha}''(t) = (0, 2, 6t)$$

$$\vec{\alpha}'''(t) = (0, 0, 6)$$

and so

$$\vec{\alpha}' \times \vec{\alpha}'' = (6t^2, -6t, 2) \quad \langle \vec{\alpha}', \vec{\alpha}'' \rangle = 18t^3 + 4t$$

$$\vec{\alpha}'' \times \vec{\alpha}''' = (12, 0, 0)$$

$$\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle = 12.$$

we also compute

$$\| \vec{\alpha}' \times \vec{\alpha}'' \| = \sqrt{36t^4 + 36t^2 + 4} = 2\sqrt{9t^4 + 9t^2 + 1}$$

$$\| \vec{\alpha}' \| = \sqrt{1 + 4t^2 + 9t^4}$$

With all of these formulae worked out, we just assemble them.

After some algebra, we get

$$\kappa(t) = \frac{2\sqrt{9t^4+9t^2+1}}{(1+4t^2+9t^4)^{3/2}}$$

$$\tau(t) = \frac{3}{1+9t^2+9t^4}$$

while

$$T = \frac{1}{\sqrt{1+4t^2+9t^4}} \begin{pmatrix} 1 \\ 2t \\ 3t^2 \end{pmatrix}$$

$$N = \frac{1}{\sqrt{1+4t^2+9t^4}\sqrt{1+9t^2+9t^4}} \begin{pmatrix} -t(2+9t^2) \\ 1-9t^4 \\ 3(t+2t^3) \end{pmatrix}$$

$$B = \frac{1}{\sqrt{1+9t^2+9t^4}} \begin{pmatrix} 3t^2 \\ -3t \\ 1 \end{pmatrix}$$