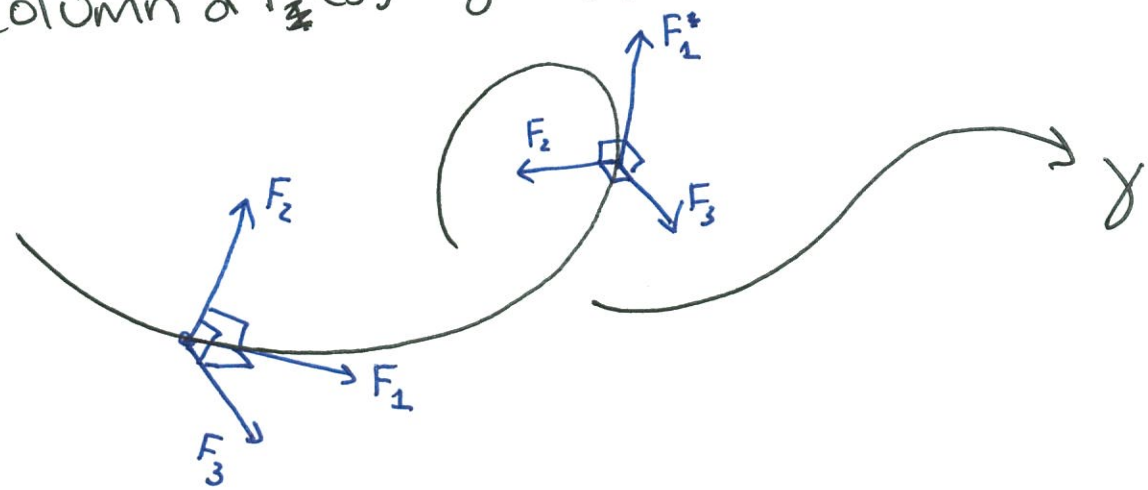
Framed Curves and Frenet Frame

We have shown that every regular curve can be reparametrized by arclength, so for the moment we don't lose anything by focusing on arclength parametrized curves.

Definition. A <u>framing</u> of $\gamma(s)$ is a map $F: \mathbb{R} \to SO(3)$ so that the first column of $F(s) = \gamma'(s)$.



Proposition. Given $F : \mathbb{R} \to SO(3)$, we can always write $F' = FS$, where S is skew-symmetric.

Proof. Since F is invertible, we can solve for $S = F^{-1}F'$, which is $F^T F'$, because F is orthogonal. Now

$$F^T F = I, \quad \text{so} \quad \frac{d}{ds}(F^T F) = 0,$$

or

$$(F')^T F + F^T (F') = 0$$

or

$$(F^T F')^T + F^T F' = 0$$

so $F^T F'$ is skew-symmetric, as promised. $\square$

We call the tangent vector

$$\gamma'(s) = T(s) = F_1(s)$$

and the remaining frame vectors $F_2(s)$ and $F_3(s)$.

We have just shown that

$$\begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T' & F_2' & F_3' \\ \downarrow & \downarrow & \downarrow \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ T & F_2 & F_3 \\ \downarrow & \downarrow & \downarrow \end{bmatrix} \begin{bmatrix} 0 & \alpha_{12} & \alpha_{13} \\ -\alpha_{12} & 0 & \alpha_{23} \\ -\alpha_{13} & -\alpha_{23} & 0 \end{bmatrix}$$

and our theory will (basically) be about finding the coefficient functions $\alpha_{12}, \alpha_{13}, \alpha_{23}$.

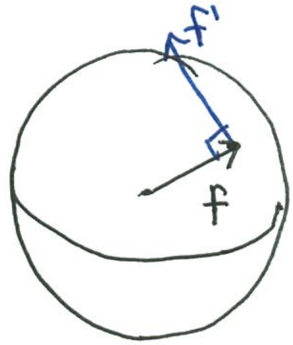
We now work out a classical example of such a construction.

Lemma. If $f, g : (a,b) \to \mathbb{R}^3$ are differentiable and $\langle f, g \rangle = \text{const}$, then $\langle f', g \rangle = -\langle f, g' \rangle$. In particular, $|f|$ const $\Leftrightarrow \langle f, f' \rangle \equiv 0$.

identically, or for all values of the parameter.

Proof. $\frac{d}{ds}\langle f, g \rangle = \langle f', g \rangle + \langle f, g' \rangle = \frac{d}{ds} \text{const} = 0.$

or



if f stays on sphere, f' is tangent to sphere and $\perp$ to f.

Now we construct our frame.
We start with $F_1(s) = T(s)$, which is

$$T(s) = \gamma'(s)$$

and let the second vector $F_2(s)$ be given by

$$N(s) = \frac{\gamma''(s)}{|\gamma''(s)|},$$ the <u>normal</u> vector

We call $|\gamma''(s)| = \kappa(s)$ the <u>curvature</u> of γ, and observe that by construction

$$T'(s) = \gamma''(s) = \kappa(s) N(s)$$

This means $\alpha_{12} = \kappa(s)$, and $\alpha_{13} = 0$. Now the third vector $F_3(s)$ must be given by $T(s) \times N(s)$ since $F \in SO(3)$. So

$$B(s) = T(s) \times N(s),$$ the <u>binormal</u>

completes the frame.

Now $\begin{bmatrix} T' & N' & B' \end{bmatrix} = \begin{bmatrix} T & N & B \end{bmatrix} \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \alpha_{23} \\ 0 & -\alpha_{23} & 0 \end{bmatrix}$, so

$$N'(s) = -\kappa(s)\,T(s) + \alpha_{23}(s)\,B(s).$$

We call $\alpha_{23}(s) = \langle N'(s), B(s) \rangle$ the <u>torsion</u> $\tau(s)$ of γ. This means that

$$B'(s) = -\tau(s)\,N(s),$$

completing the <u>Frenet equations</u>.

$$\boxed{\begin{aligned} T'(s) &= \;\; \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) \qquad\qquad + \tau(s)B(s) \\ B'(s) &= \; -\tau(s)N(s) \end{aligned}}$$

This frame is called the <u>Frenet frame</u>.

Example. Find the Frenet frame, curvature, and torsion of the helix

$$\gamma(s) = \left(a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} \right).$$

We start with

$$T(s) = \gamma'(s) = \left(-a\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right)\cdot\frac{1}{\sqrt{a^2+b^2}},\right.$$
$$a\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right)\frac{1}{\sqrt{a^2+b^2}},$$
$$\left.\frac{b}{\sqrt{a^2+b^2}}\right)$$

(It's easy to check $|T(s)|^2 = \frac{a^2+b^2}{a^2+b^2} = 1$.)

Now

$$T'(s) = \left(\frac{-a}{a^2+b^2}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right),\right.$$
$$\frac{-a}{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right),$$
$$\left.0\right)$$
$$= K(s)\,N(s).$$

A quick way to compute $K(s)$ is to take the norm of $T'(s)$.

Doing so, we get

$$|T'|^2 = \cancel{\frac{a^2\cos^2}{(a^2+b^2)^2} + \frac{a^2}{(a^2+b^2)^2}} = \frac{\cancel{2}a^2}{(a^2+b^2)^2},$$

so we have

$$|T'| = \kappa(s) = \frac{\cancel{2}a}{a^2+b^2}.$$

This means

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), -\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), 0\right)$$

We now find $B(s)$ by taking $T(s) \times N(s)$.

two - signs!

$$B(s) = \Bigg(0 + \frac{b}{\sqrt{a^2+b^2}} \cdot \sin\left(\frac{s}{\sqrt{a^2+b^2}}\right), \cancel{\frac{b}{\sqrt{a^2+b^2}} \cdot -\cos(}$$

$$\frac{b}{\sqrt{a^2+b^2}} \cdot \cancel{\frac{-a}{a^2+b^2}} \cos\left(\frac{s}{\sqrt{a^2+b^2}}\right) - 0,$$

$$\cancel{-\frac{a}{\sqrt{a^2+b^2}} \sin^2\left(\frac{s}{\sqrt{a^2+b^2}}\right) - \frac{a}{\sqrt{a^2+b^2}} \cos^2\left(\frac{s}{\sqrt{a^2+b^2}}\right)}$$

$$\frac{a}{\sqrt{a^2+b^2}}\Bigg)$$ ← signs do work out this way.

Now it's easiest to find τ ~~τ~~ by taking

$$B'(s) = \left(\frac{b}{a^2+b^2}\cos\left(\frac{s}{\sqrt{a^2+b^2}}\right), \right.$$
$$\frac{b}{a^2+b^2}\sin\left(\frac{s}{\sqrt{a^2+b^2}}\right),$$
$$\left. 0 \right)$$

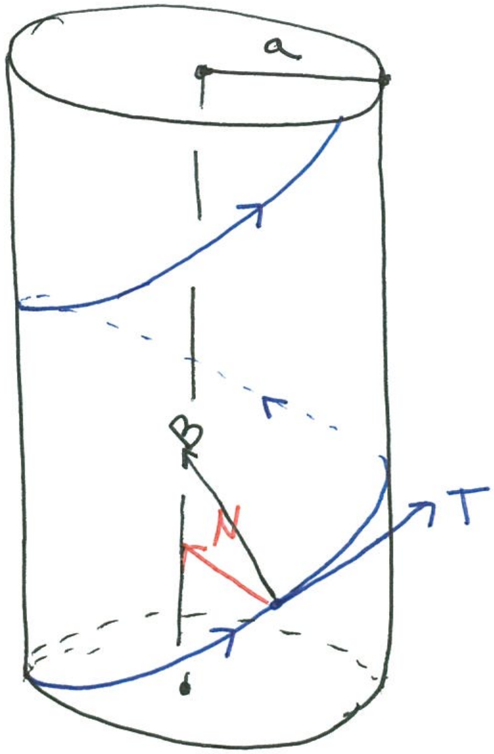
and using $\tau(s) = B'(s)\cdot N(s)$. Note that unlike curvature, torsion can be negative, so it's not ok to just take the norm of $B'(s)$. We get

$$\tau(s) = \frac{b}{a^2+b^2}.$$

We see that the helix has constant curvature and constant torsion.

Exercise (hwk) Show that if γ has constant curvature and torsion, then γ is a helix.

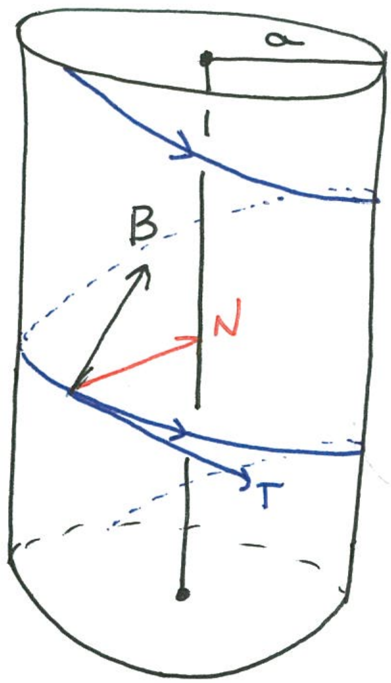
The pictures are interesting:



positive torsion
helix is a right-handed screw.

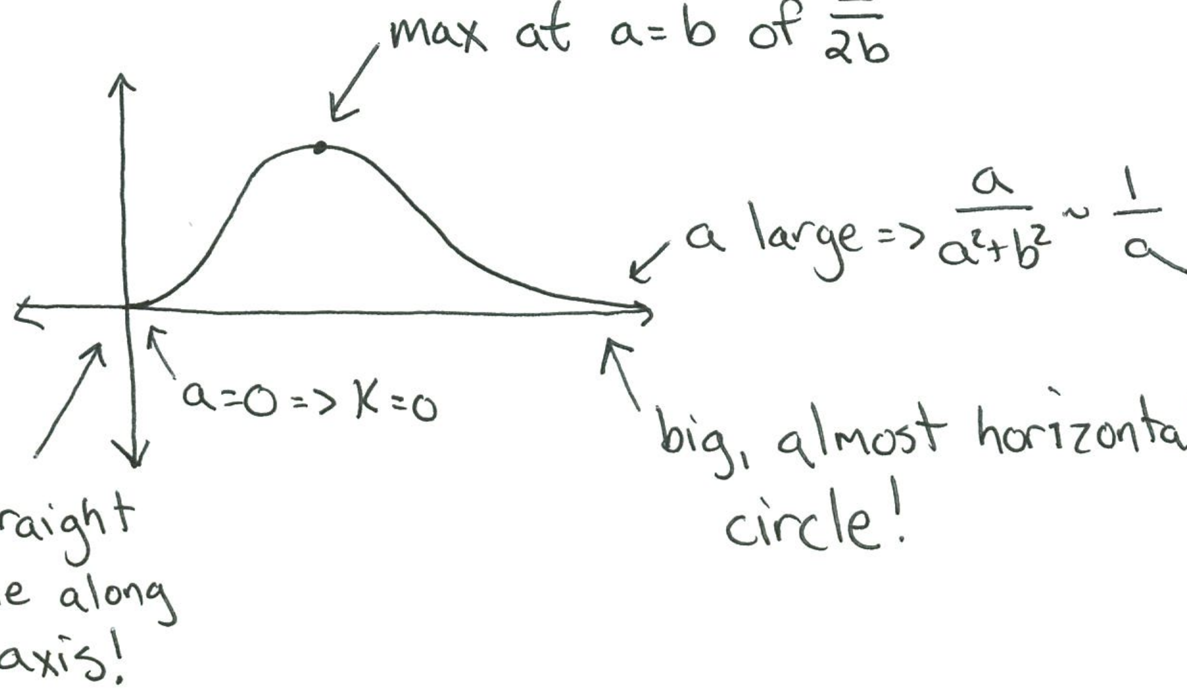
N always points to center axis of cylinder.

B points up



negative torsion helix is a left handed screw, normal still points to central axis B still points up.

If we hold b constant and vary the radius of the cylinder a, we see



This has a neat practical application. Many cables and chains have a fixed maximum curvature - there is only so tightly they bend. Putting tension on pulls them toward the central axis (trust me!).