A Length-Rescaled Curvature Flow

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1 Preliminary Results

We define a new curvature flow which deforms plane curves along their unit normal vectors at a rate proportional to the local curvature, such that the total length of the curve cannot change:

$$X_t = \kappa N + \left(\frac{1}{L} \int \kappa^2 ds \right) X \tag{1}$$

where X is the parametrization of a plane curve with arc-length parameter s, κ denotes signed curvature, N is the indward-pointing unit normal, and L is the total length of the curve. Let T be the parametrization of the unit tangent to the curve. As defined, our flow differs from that developed by M. Gage, R. Hamilton, M. Grayson, and others in the addition of the second term in (1). We shall proceed to show that the total length of the curve now remains fixed, and we shall explore some of the initial results for the new flow, taking cues from Gage [1]. Henceforth, let the curve parametrization be denoted $X(\phi, t)$, where ϕ is the curve parameter and t is a 'time' parameter distinguishing members of the family of curves generated during the deformation.

1.1 Length-Rescaling

Recall that the total length of a plane curve is given by

$$L = \int \left| \frac{\partial X}{\partial \phi} \right| d\phi \tag{2}$$

Hence, the time derivative of the total length of a curve is given by

$$L_t = \int \left| \frac{\partial X}{\partial \phi} \right|_t d\phi \tag{3}$$

And we are therefore interested in the t-dependence of the integrand. Notice that:

[1] M. Gage, *An Isopermietric Inequality with Applications to Curve Shortening*, Duke Mathematical Journal 50 (1983), 1225-1229.

$$\begin{aligned}
\left|\frac{\partial X}{\partial \phi}\right|_t &= \frac{\partial}{\partial t}\left(\left\langle \frac{\partial X}{\partial \phi}, \frac{\partial X}{\partial \phi}\right\rangle^{1/2}\right) = \frac{\left\langle \frac{\partial X}{\partial \phi}, \frac{\partial^2 X}{\partial t \partial \phi}\right\rangle}{\left|\frac{\partial X}{\partial \phi}\right|} = \frac{\left\langle \frac{\partial X}{\partial \phi}, \frac{\partial^2 X}{\partial \phi \partial t}\right\rangle}{\left|\frac{\partial X}{\partial \phi}\right|} \\
&= \left\langle T, \frac{\partial}{\partial \phi}\left(\kappa N + \frac{X}{L}\int \kappa^2 ds\right)\right\rangle = \left\langle T, \frac{\partial \kappa}{\partial \phi} N - \kappa^2 \left|\frac{\partial X}{\partial \phi}\right| T + \frac{T}{L}\left|\frac{\partial X}{\partial \phi}\right| \int \kappa^2 ds\right\rangle \\
&= \left(\frac{1}{L}\int \kappa^2 ds\right)\left|\frac{\partial X}{\partial \phi}\right| - \kappa^2 \left|\frac{\partial X}{\partial \phi}\right|
\end{aligned}$$

where we have made use of the Frenet formulas and the orthogonality of T and N. If we now integrate this result with respect to ϕ, we obtain an expression for (3):

$$L_t = \left(\frac{1}{L}\int \kappa^2 ds\right)\int \left|\frac{\partial X}{\partial \phi}\right| d\phi - \int \kappa^2 \left|\frac{\partial X}{\partial \phi}\right| d\phi = \frac{L}{L}\int \kappa^2 ds - \int \kappa^2 ds = 0$$

since $\left|\frac{\partial X}{\partial \phi}\right| d\phi = ds$ and $\int ds$ along the whole curve is just the total length L.

Hence, we see that the total length of the curve does not change under this modified flow. Note, however, that this is a condition which arises from integration and which will not hold locally.

1.2 Area

As Gage shows, we can use the support function $p = -\langle X, N\rangle$ to express the area enclosed by a plane curve:

$$2A = -\int \langle X, N\rangle \left|\frac{\partial X}{\partial \phi}\right| d\phi \tag{4}$$

Then, as before, we differentiate both sides with respect to t:

$$\begin{aligned}
2A_t &= -\int \left\langle \kappa N + \left(\frac{1}{L}\int \kappa^2 ds\right) X, N\right\rangle \left|\frac{\partial X}{\partial \phi}\right| d\phi - \int \langle X, N_t\rangle \left|\frac{\partial X}{\partial \phi}\right| d\phi \\
&\quad - \int \langle X, N\rangle \left(\frac{1}{L}\left|\frac{\partial X}{\partial \phi}\right| \int \kappa^2 ds - \kappa^2 \left|\frac{\partial X}{\partial \phi}\right|\right) d\phi
\end{aligned}$$

To simplify this expression, we'll need to focus on the N_t term. To begin with, we differentiate (1) with respect to ϕ, which we have already done in deriving an expression for $\left|\frac{\partial X}{\partial \phi}\right|_t$:

$$X_{t\phi} = \frac{\partial \kappa}{\partial \phi} N - \kappa^2 \left|\frac{\partial X}{\partial \phi}\right| T - \left(\frac{1}{L}\left|\frac{\partial X}{\partial \phi}\right| \int \kappa^2 ds\right) T = X_{\phi t} = \left(T\left|\frac{\partial X}{\partial \phi}\right|\right)_t \tag{5}$$

Now we take the inner product of the terms in (5) with N, obtaining:

$$\left\langle \left(T \left| \frac{\partial X}{\partial \phi} \right| \right)_t, N \right\rangle = -\left\langle T \left| \frac{\partial X}{\partial \phi} \right|, N_t \right\rangle = \frac{\partial \kappa}{\partial \phi} \tag{6}$$

and it therefore follows that

$$N_t = -\frac{\frac{\partial \kappa}{\partial \phi}}{\left| \frac{\partial X}{\partial \phi} \right|} T \tag{7}$$

Returning to the time derivative of area, we can now write

$$\begin{aligned} 2A_t &= -\int \left\langle \kappa N + \left(\frac{1}{L} \int \kappa^2 ds \right) X, N \right\rangle \left| \frac{\partial X}{\partial \phi} \right| d\phi - \int \left\langle X, -\frac{\frac{\partial \kappa}{\partial \phi}}{\left| \frac{\partial X}{\partial \phi} \right|} T \right\rangle \left| \frac{\partial X}{\partial \phi} \right| d\phi \\ &\quad - \int \langle X, N \rangle \left(\frac{1}{L} \left| \frac{\partial X}{\partial \phi} \right| \int \kappa^2 ds - \kappa^2 \left| \frac{\partial X}{\partial \phi} \right| \right) d\phi \\ &= -2\pi - \left(\frac{2}{L} \int \kappa^2 ds \right) \int \langle X, N \rangle ds + \int \frac{\partial \kappa}{\partial \phi} \langle X, T \rangle d\phi + \int \kappa^2 \langle X, N \rangle ds \end{aligned}$$

Now, we integrate the third term by parts in order to proceed. Move the derivative from $\frac{\partial \kappa}{\partial \phi}$ to the inner product, so that $\int \frac{\partial \kappa}{\partial \phi} \langle X, T \rangle d\phi = \kappa \langle X, T \rangle - \int \kappa ds - \int \kappa^2 \langle X, N \rangle ds$, but the first term should vanish since it will be the same at the endpoints. Furthermore, we recall that, around the closed curve, $\int \kappa ds = 2\pi$. Recalling also the relationship we stated earlier between the support function and area, we now have:

$$A_t = -2\pi - \left(\frac{1}{L} \int \kappa^2 ds \right) \int \langle X, N \rangle ds = -2\pi + \left(\frac{2}{L} \int \kappa^2 ds \right) A \tag{8}$$

as our final expression for the time derivative of area under the length-rescaled flow.

1.3 Curvature

To obtain an expression for the time derivative of curvature, we begin by differentiating our expression for N_t (7) with respect to ϕ.

$$N_{t\phi} = -\frac{\partial^2 \kappa}{\partial \phi^2} \left| \frac{\partial X}{\partial \phi} \right|^{-1} T + \frac{\partial \kappa}{\partial \phi} \left| \frac{\partial X}{\partial \phi} \right|^{-2} \left| \frac{\partial X}{\partial \phi} \right|_\phi T - \frac{\partial \kappa}{\partial \phi} \kappa N = N_{\phi t} = \left(-\left| \frac{\partial X}{\partial \phi} \right| \kappa T \right)_t \tag{9}$$

This last term is

$$\left(-\left| \frac{\partial X}{\partial \phi} \right| \kappa T \right)_t = -\left| \frac{\partial X}{\partial \phi} \right|_t \kappa T - \left| \frac{\partial X}{\partial \phi} \right| \kappa_t T - \left| \frac{\partial X}{\partial \phi} \right| \kappa T_t \tag{10}$$

and so we shall need an expression for T_t. In (6) we showed that $\left\langle \left(T \left| \frac{\partial X}{\partial \phi} \right| \right)_t, N \right\rangle = \frac{\partial \kappa}{\partial \phi}$. By carrying out the indicated differentiation and plugging in the expression we obtained for $\left| \frac{\partial X}{\partial \phi} \right|_t$, we find that

$$T_t = \frac{\partial \kappa}{\partial \phi} \left| \frac{\partial X}{\partial \phi} \right|^{-1} N \tag{11}$$

Now (9) becomes

$$\left(- \left| \frac{\partial X}{\partial \phi} \right| \kappa T \right)_t = - \left| \frac{\partial X}{\partial \phi} \right|_t \kappa T - \left| \frac{\partial X}{\partial \phi} \right| \kappa_t T - \kappa \frac{\partial \kappa}{\partial \phi} N$$

Putting this into (9) and then dotting that entire expression with T yields

$$-\frac{\partial^2 \kappa}{\partial \phi^2} \left| \frac{\partial X}{\partial \phi} \right|^{-1} + \frac{\partial \kappa}{\partial \phi} \left| \frac{\partial X}{\partial \phi} \right|^{-2} \left| \frac{\partial X}{\partial \phi} \right|_\phi = - \left| \frac{\partial X}{\partial \phi} \right|_t \kappa - \left| \frac{\partial X}{\partial \phi} \right| \kappa_t \tag{12}$$

which we can rewrite as

$$\kappa_t = \left| \frac{\partial X}{\partial \phi} \right|^{-1} \frac{\partial}{\partial \phi} \left(\frac{\partial \kappa}{\partial \phi} \left| \frac{\partial X}{\partial \phi} \right|^{-1} \right) - \left| \frac{\partial X}{\partial \phi} \right|^{-1} \left| \frac{\partial X}{\partial \phi} \right|_t \kappa \tag{13}$$

and so plugging in the expression for $\left| \frac{\partial X}{\partial \phi} \right|_t$

$$\kappa_t = \left| \frac{\partial X}{\partial \phi} \right|^{-1} \frac{\partial}{\partial \phi} \left(\frac{\partial \kappa}{\partial \phi} \left| \frac{\partial X}{\partial \phi} \right|^{-1} \right) + \kappa^3 - \frac{\kappa}{L} \int \kappa^2 ds \tag{14}$$

Note that this differs from the time derivative of curvature for the ordinary curve-shortening flow only in the last term. Without this term, Gage[2] remarks that one can use the maximum principle to deduce that if curvature is initially everywhere positive, it will remain so under the flow.

1.4 The Isoperimetric Ratio

We are now ready to state a theorem.

Theorem 1 *A family of C^2, convex curves $\gamma(t)$ satisfying the evolution equation* (1) *for* $0 < t < T$ *also satisfies*

$$\lim_{t \to T} \frac{L}{4\pi A} = 1$$

Proof. We begin by computing the time derivative of the isoperimetric ratio using the time derivatives of length and area. So:

$$\left(\frac{L^2}{4\pi A} \right)_t = \frac{L L_t}{2\pi A} - \frac{L^2}{4\pi A^2} A_t = -\frac{L^2}{4\pi A^2} A_t \tag{15}$$

and with (8) this becomes

$$\left(\frac{L^2}{4\pi A} \right)_t = \frac{L^2}{2A^2} - \frac{L}{2\pi A} \int \kappa^2 ds \tag{16}$$

[2] M. Gage, *Curve Shortening Makes Convex Curves Circular*, Invent. Math. 76 (1984), 357-364.

What can we say about the sign of this quantity? Setting the expression less than zero and rearranging, we obtain the condition

$$\int \kappa^2 ds > \pi \frac{L}{A} \tag{17}$$

Gage has shown in Lemma 3, page 359 of [2] that there is a nonnegative functional $F(\gamma)$ which is defined for all convex, C^2 curves γ and which satisfies

$$\left(\int \kappa^2 ds \right) (1 - F(\gamma)) - \pi \frac{L}{A} \geq 0 \tag{18}$$

and $F(\gamma) = 0$ when γ is a circle. Hence, (17) must be satisfied for all convex, C^2 curves, and therefore the time derivative of the isoperimetric ratio is always negative under the evolution (1) unless the curve is a circle.

We conclude that the isoperimetric ratio for C^2, convex curves is strictly decreasing until it reaches the value 1, at which point the curve is a circle and the ratio no longer changes. □

We shall now consider some results of a different nature for this length-rescaled flow. The notation may change slightly in places in order to correspond more closely with the references we'll be using.

2 The Chord Length to Arc Length Ratio for Open Curves Undergoing Length-Rescaled Curvature Flow

Theorem 2 *Let d and l denote the chord length and arc length, respectively, between any two points p and q on an open curve, so that*

$$d = |X(p,t) - X(q,t)| \tag{19}$$

and

$$l = \int_p^q \left|\frac{\partial X}{\partial \phi}\right| d\phi \tag{20}$$

Then let $X : \Gamma \times [0,T] \to \boldsymbol{R}^2$ be an embedded solution of the length-rescaled curvature flow (1), where $\Gamma \neq S^1$ so that l is smoothly defined on $\Gamma \times \Gamma$. Then the minimum of d/l on Γ is nondecreasing; it is strictly increasing unless $d/l \equiv 1$ and Γ is a straight line segment.

Proof. This theorem, for ordinary curve-shortening flow, is due to G. Huisken[3]. We follow his methods of proof, adapting them for the new flow. Furthermore, many of his calculations are suppressed, and we shall here try to be very explicit.

Since d and l are smooth functions off the diagonal of $\Gamma \times \Gamma$, it suffices to show that whenever their ratio attains a spatial minimum for some pair of points $(p,q) \in \Gamma \times \Gamma$ at some time $t_0 \in [0,T]$, we have

$$\frac{d}{dt}\left(\frac{d}{l}\right)(p,q,t_0) \geq 0 \tag{21}$$

Assume without loss of generality that $p \neq q$ and that if s is the arclength parameter at t_0 then $s(p) \geq s(q)$ at t_0. Then, by the assumption of a spatial minimum we have that the first and second 'variations' (for our purposes these will simply be directional derivatives) obey the following:

$$\delta(\xi)\left(\frac{d}{l}\right) = 0 \tag{22}$$

$$\delta^2(\xi)\left(\frac{d}{l}\right) \geq 0 \tag{23}$$

for vectors $\xi \in T_p\Gamma_{t_0} \oplus T_q\Gamma_{t_0}$, the space of tangent vectors at p and q at time t_0.

It will be helpful to re-parametrize the curve locally around p and q using arc-length parameters u and v, respectively, so that the curve is described at time t_0 near these points by the parametrizations $X(u,t_0)$ and $X(v,t_0)$. Then we shall need to define several vectors before continuing. Have e_1 and e_2 denote the unit tangent vectors along the curve at p and q:

$$e_1 = \frac{\partial X(u,t_0)}{\partial u} \tag{24}$$

[3] G. Huisken, *A Distance Comparison Principle for Evolving Curves*, Asian J. Math. 2 (1998), 127-134

$$e_2 = \frac{\partial X(v, t_0)}{\partial v} \tag{25}$$

Let ω denote the unit vector in the direction from p to q:

$$\omega = \frac{X(v,t) - X(u,t)}{d} \tag{26}$$

and then note that our first variation obeys a Leibniz rule

$$\delta\left(\frac{d}{l}\right) = \frac{\delta(d)}{l} - \frac{d}{l^2}\delta(l) \tag{27}$$

so that we need only compute the variations of d and l individually. In order to compute the first variation of d, we shall need to first compute d_u, d_v, and d_t. We pause now to do so. Let the curvature vector $\vec{\kappa}$ stand for $\kappa\mathbf{N}$ in what follows.

$$d_u = \frac{\langle X(u,t) - X(v,t), e_1\rangle}{d} = -\langle \omega, e_1\rangle \tag{28}$$

$$d_v = \frac{\langle X(u,t) - X(v,t), -e_2\rangle}{d} = \langle \omega, e_2\rangle \tag{29}$$

$$\begin{aligned} d_t &= \frac{1}{d}\left\langle (X(u,t) - X(v,t)), \vec{\kappa}(u,t) + \left(\frac{1}{L}\int \kappa^2 ds\right) X(u,t) - \vec{\kappa}(v,t) - \left(\frac{1}{L}\int \kappa^2 ds\right) X(v,t)\right\rangle \\ &= \langle -\omega, \vec{\kappa}(u,t) - \vec{\kappa}(v,t)\rangle - \left(\frac{1}{L}\int \kappa^2 ds\right)\langle \omega, X(u,t) - X(v,t)\rangle \\ &= \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{L}\int \kappa^2 ds \end{aligned}$$

We now consider the vanishing of the first variation along e_1 and e_2. We first compute the first variations of d and l in these directions and then proceed to plug them into (27).

$$\delta(e_1 \oplus 0)(d) = D_{e_1} d = \langle e_1, \nabla d\rangle = d_u = -\langle \omega, e_1\rangle \tag{30}$$

$$\delta(e_1 \oplus 0)(l) = -1 \tag{31}$$

$$\delta(0 \oplus e_2)(d) = D_{e_2} d = \langle e_2, \nabla d\rangle = d_v = \langle \omega, e_2\rangle \tag{32}$$

$$\delta(0 \oplus e_2)(l) = 1 \tag{33}$$

and so, plugging these into (27), we get

$$\delta(e_1 \oplus 0)\left(\frac{d}{l}\right) = \frac{d}{l^2} - \frac{\langle \omega, e_1\rangle}{l} = 0 \tag{34}$$

$$\delta(0 \oplus e_2)\left(\frac{d}{l}\right) = \frac{\langle \omega, e_2 \rangle}{l} - \frac{d}{l^2} = 0 \tag{35}$$

from which it follows that

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l} \tag{36}$$

which we'll want to keep in mind for future calculations.

Now we turn to the second variation, for which we can write

$$\delta^2\left(\frac{d}{l}\right) = \frac{\delta^2(d)}{l} - 2\frac{\delta(d)\delta(l)}{l^2} + 2\frac{d(\delta(l))^2}{l^3} - \delta^2(l)\frac{d}{l^2} \geq 0 \tag{37}$$

Here, we must now consider two cases.

2.0.1 Case 1: $e_1 = e_2$

Here, we are essentially moving in the same direction along the curve at p and at q, and so all variations of l now vanish, and we need only consider the first term in equation (37). Computing the second variation of d:

$$\delta^2(e_1 \oplus e_2)(d) = \left\langle H(d)\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle = d_{uu} + 2d_{uv} + d_{vv} \tag{38}$$

where $H(d)$ was the Hessian matrix of partial derivatives of d. Now, by differentiating (28) and (29) and using (36), it should turn out that

$$d_{uu} = \frac{1}{d} - \frac{d}{l^2} - \langle \omega, \vec{\kappa}(u,t) \rangle \tag{39}$$

$$d_{vv} = \frac{1}{d} - \frac{d}{l^2} + \langle \omega, \vec{\kappa}(v,t) \rangle \tag{40}$$

$$d_{uv} = \frac{d}{l^2} - \frac{1}{d} \tag{41}$$

Plugging these into (38) and then (37), we get

$$\delta^2(e_1 \oplus e_2)\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(e_1 \oplus e_2)(d) = \frac{1}{l}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \geq 0 \tag{42}$$

2.0.2 Case 2: $e_1 \neq e_2$

We shall now choose $\xi = e_1 \ominus e_2$ so that

$$\delta(e_1 \ominus e_2)(l) = -2 \tag{43}$$

and

$$\delta(e_1 \ominus e_2)(d) = -\langle \omega, e_1 + e_2 \rangle \tag{44}$$

We perform calculations analogous to those in Case 1. We know from before, or can easily compute, the following:

$$\delta(e_1 \ominus e_2)\left(\frac{d}{l}\right) = 2\frac{d}{l} - \frac{1}{l}\langle \omega, e_1 + e_2 \rangle = 0 \tag{45}$$

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{l} \tag{46}$$

$$\delta^2\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(d) - \frac{2\delta(d)\delta(l)}{l^2} + \frac{2d(\delta(l))^2}{l^3} - \frac{d}{l^2}\delta^2(l) \geq 0 \tag{47}$$

Note that

$$\langle e_1 + e_2, e_1 + e_2 \rangle = |e_1 + e_2|^2 \quad \Rightarrow \quad 2\langle e_1, e_2 \rangle = |e_1 + e_2|^2 - 2 \tag{48}$$

which we shall need later on. Now, plugging equations (43) and (44) into equation (47) , we can simplify that inequality to

$$\delta^2\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(d) - \frac{4}{l^2}\langle \omega, e_1 + e_2 \rangle + 8\frac{d}{l^3} \geq 0 \tag{49}$$

Using (46), we simplify the middle term as follows

$$\delta^2\left(\frac{d}{l}\right) = \frac{1}{l}\delta^2(d) - \frac{4}{l^2}\left(2\frac{d}{l}\right) + 8\frac{d}{l^3} = \frac{1}{l}\delta^2(d) \geq 0 \tag{50}$$

and so we need only compute $\delta^2(d)$, which will not be the same as in Case 1. There are two reasons for this: for one, our variational vector is now $(1, -1)$. Furthermore, we will get cross-terms when we compute d_{uv}, which is why we needed an expression for $\langle e_1, e_2 \rangle$. In the first case, this was simply 1.

Now,

$$\delta^2(e_1 \ominus e_2)(d) = \left\langle H(d)\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = d_{uu} - 2d_{uv} + d_{vv} \tag{51}$$

and so we need to compute these second partials. Starting with $d_u = -\langle \omega, e_1 \rangle$ and $d_v = \langle \omega, e_2 \rangle$, we have

$$d_{uu} = -\langle \omega_u, e_1 \rangle - \langle \omega, \vec{\kappa}(u, t) \rangle \tag{52}$$

and so we need to compute ω_u:

$$\omega_u = -\frac{1}{d}e_1 - \frac{X(v,t) - X(u,t)}{d^2}d_u = \omega\frac{\langle\omega, e_1\rangle}{d} - \frac{e_1}{d} \tag{53}$$

With this in hand, and using (46), (52) becomes

$$d_{uu} = \frac{1}{d} - \frac{d}{l^2} - \langle\omega, \vec{\kappa}(u,t)\rangle \tag{54}$$

Following the same procedure, we can compute

$$d_{vv} = \frac{1}{d} - \frac{d}{l^2} + \langle\omega, \vec{\kappa}(v,t)\rangle \tag{55}$$

Lastly, using equation (48) we can compute

$$d_{uv} = -\langle\omega_v, e_1\rangle = \frac{d}{l^2} - \frac{1}{d}\langle e_1, e_2\rangle = \frac{d}{l^2} - \frac{1}{2d}|e_1 + e_2|^2 + \frac{1}{d} \tag{56}$$

Putting these three results back into (51), we should get

$$\delta^2(e_1 \ominus e_2)\left(\frac{d}{l}\right) = \frac{1}{dl}|e_1 + e_2|^2 - 4\frac{d}{l^3} + \frac{1}{l}\langle\omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle \geq 0 \tag{57}$$

But we shall show that the first two terms are, in fact, equal. For $\xi = e_1 \ominus e_2$, $\omega||(e_1 + e_2)$, and so we can use the fact that $\langle\omega, e_1 + e_2\rangle = |e_1 + e_2|$ to rewrite the first term above:

$$\frac{1}{dl}\langle\omega, e_1 + e_2\rangle^2 = \frac{1}{dl}(\langle\omega, e_1\rangle + \langle\omega, e_2\rangle)^2 = \frac{1}{dl}\left(\frac{d}{l} + \frac{d}{l}\right) = 4\frac{d}{l^3} \tag{58}$$

so the first two terms of (57) cancel and we're left with the same conclusion as in Case 1, as we hoped:

$$\frac{1}{l}\langle\omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle \geq 0 \tag{59}$$

We are now ready to consider the time derivative of the original quantity of interest, d/l. With what we already know, we can go ahead and write

$$\left(\frac{d}{l}\right)_t = \frac{d_t}{l} - \frac{d}{l^2}l_t = \frac{1}{l}\langle\omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{lL}\int \kappa^2 ds - \frac{d}{l^2}l_t \tag{60}$$

In order to say more about this result we shall need to come up with an expression for l_t. Remember: while the *total* length of the curve does not change, we do not know what is happening locally, and l refers only to the arc length between two particular points p and q. Recall, to begin with, that we can express this time derivative as $l_t = \int_p^q \left|\frac{\partial X}{\partial\phi}\right|_t d\phi$. From page 2, we have seen that we can express the integrand here as $\left(\frac{1}{L}\int \kappa^2 ds\right)\left|\frac{\partial X}{\partial \phi}\right| - \kappa^2 \left|\frac{\partial X}{\partial \phi}\right|$. Integrating this with respect to ϕ, now from p to q, we'll see that

$$l_t = \frac{l}{L}\int \kappa^2 ds - \int_p^q \kappa^2 ds \tag{61}$$

Returning now to (60) with this result (which causes some cancellation), and using the results obtained in (42) and (59), we see that:

$$\left(\frac{d}{l}\right)_t = \frac{1}{l}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{l^2}\int_p^q \kappa^2 ds \geq \frac{d}{l^2}\int_p^q \kappa^2 ds \tag{62}$$

This last term is obviously greater than zero off the diagonal of $\Gamma \times \Gamma$, where it is identically zero. Hence,

$$\left(\frac{d}{l}\right)_t \geq \frac{d}{l^2}\int_p^q \kappa^2 ds \geq 0 \tag{63}$$

and the theorem has been proven. □

3 The Chord Length to Arc Length Ratio for Closed Curves Undergoing Length-Rescaled Curvature Flow

We use the same notation as in the proof of the theorem for open curves. However, we are now going to want to define a new quantity since l is no longer smoothly defined for closed curves. Let $\psi : S^1 \times S^1 \times [0,T] \to \mathbf{R}$ be given by

$$\psi \equiv \frac{L}{\pi} \sin\left(\frac{l\pi}{L}\right) \tag{64}$$

where l is the arc length between two points on the curve, as before, and L is the total length of the curve.

Theorem 3 *Let $X : S^1 \times [0,T] \to \boldsymbol{R}^2$ be a smooth solution of the length-rescaled curvature flow (1). Then the minimum of d/ψ is nondecreasing; it is strictly increasing unless $d/\psi \equiv 1$ and $X(S^1)$ is a round circle.*

Proof. As before, the theorem and proof follow closely those of Gerhard Huisken. And, as before, we shall attempt to be as explicit as possible in our computations.

So once again, it suffices to show that whenever d/ψ attains a spatial minimum for some pair of points $(p,q) \in S^1 \times S^1$ at some time $t_0 \in [0,T]$, then

$$\frac{d}{dt}\left(\frac{d}{\psi}\right)(p,q,t_0) \geq 0 \tag{65}$$

Let s once again be the arclength parameter at t_0, and assume $0 \leq s(p) \leq s(q) \leq \frac{1}{2}L(t_0)$, so that $l(p,q,t_0) = s(q) - s(p)$. We shall reprise Huisken's 'variational' methods. So, by assumption:

$$\delta(\xi)\left(\frac{d}{\psi}\right)(p,q,t_0) = 0 \tag{66}$$

$$\delta^2(\xi)\left(\frac{d}{\psi}\right)(p,q,t_0) \geq 0 \tag{67}$$

for variations $\xi \in T_pS^1_{t_0} \oplus T_qS^1_{t_0}$. Let's first consider, just as we did before, the variations $\xi = e_1 \oplus 0$ and $\xi = 0 \oplus e_2$. Because our definitions of d and l have not changed, we can use our previous computations of the derivatives and variations in order to say that

$$\delta(e_1 \oplus 0)(d) = -\langle \omega, e_1 \rangle \tag{68}$$

$$\delta(0 \oplus e_2)(d) = \langle \omega, e_2 \rangle \tag{69}$$

$$\delta(e_1 \oplus 0)(l) = -1 \tag{70}$$

$$\delta(0 \oplus e_2)(l) = 1 \tag{71}$$

Then we can compute

$$\delta(e_1 \oplus 0)(\psi) = \frac{d}{dl}(\psi)\delta(e_1 \oplus 0)(l) = -\cos\left(\frac{l\pi}{L}\right) \tag{72}$$

$$\delta(0 \oplus e_2)(\psi) = \frac{d}{dl}(\psi)\delta(0 \oplus e_2)(l) = \cos\left(\frac{l\pi}{L}\right) \tag{73}$$

Now, remembering that $\delta(d/\psi) = \frac{\delta(d)}{\psi} - \frac{d}{\psi^2}\delta(\psi)$, we can plug in equations (68), (69), (72), and (73) to show that

$$\delta(e_1 \oplus 0)\left(\frac{d}{\psi}\right) = \frac{-\langle \omega, e_1 \rangle}{\psi} + \frac{d}{\psi^2}\delta(\psi) = 0 \tag{74}$$

$$\delta(0 \oplus e_2)\left(\frac{d}{\psi}\right) = \frac{\langle \omega, e_2 \rangle}{\psi} - \frac{d}{\psi^2}\delta(\psi) = 0 \tag{75}$$

from which it follows that

$$\langle \omega, e_1 \rangle = \langle \omega, e_2 \rangle = \frac{d}{\psi}\cos\left(\frac{l\pi}{L}\right) \tag{76}$$

which we shall want to keep in mind.

Now we consider the second variation, which satisfies

$$\delta^2\left(\frac{d}{\psi}\right) = \frac{\delta^2(d)}{\psi} - 2\frac{\delta(d)\delta(\psi)}{\psi^2} + 2\frac{d(\delta(\psi))^2}{\psi^3} - \frac{d}{\psi^2}\delta^2(\psi) \geq 0 \tag{77}$$

And once more, we consider two cases.

3.0.3 Case 1: $e_1 = e_2$

Choose $\xi = e_1 \oplus e_2$. Because variations of l vanish in this case, all variations of ψ will also vanish, since we've seen that variations of l appear on differentiation of ψ. Thus, we have reduced the problem to computing $\delta^2(e_1 \oplus e_2)\left(\frac{d}{\psi}\right) = \frac{\delta^2(d)}{\psi}$. We computed the numerator in the proof of the theorem for open curves, so we conclude that

$$\frac{1}{\psi}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \geq 0 \tag{78}$$

where again we've parametrized locally around p and q using u and v, respectively.

3.0.4 Case 2: $e_1 \neq e_2$

Choose $\xi = e_1 \ominus e_2$. Variations of l no longer vanish; now $\delta(l) = -2$. So from (77) we now have $\delta^2\left(\frac{d}{\psi}\right) = \frac{\delta^2(d)}{\psi} - 2\frac{\delta(d)\delta(\psi)}{\psi^2} + 2\frac{d(\delta(\psi))^2}{\psi^3} \geq 0$. First we shall compute $\delta^2(d)$ for this variation. Recall that

$$\delta^2(e_1 \ominus e_2)(d) = d_{uu} - 2d_{uv} + d_{vv} \tag{79}$$

We'll pause to compute these second partials, which are now different than in Case 1. Begin with

$$d_u = -\langle \omega, e_1 \rangle \tag{80}$$

$$d_v = \langle \omega, e_2 \rangle \tag{81}$$

To begin computing the second partials of these, we need to recall equation (76) and make appropriate substitutions. We should find that:

$$d_{uu} = \frac{1}{d} - \frac{d}{\psi^2}\cos^2\left(\frac{l\pi}{L}\right) - \langle \omega, \vec{\kappa}(u,t) \rangle \tag{82}$$

$$d_{vv} = \frac{1}{d} - \frac{d}{\psi^2}\cos^2\left(\frac{l\pi}{L}\right) + \langle \omega, \vec{\kappa}(v,t) \rangle \tag{83}$$

$$d_{uv} = \frac{d}{\psi^2}\cos^2\left(\frac{l\pi}{L}\right) - \frac{1}{d}\langle e_1, e_2 \rangle \tag{84}$$

Putting these into (79) and using the fact that $2\langle e_1, e_2 \rangle = |e_1 + e_2|^2 - 2$, we have

$$\delta^2(e_1 \ominus e_2)(d) = \frac{1}{d}|e_1 + e_2|^2 - 4\frac{d}{\psi^2}\cos^2\left(\frac{l\pi}{L}\right) + \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \tag{85}$$

Now, in this case, since $\omega || (e_1 + e_2)$, we can rewrite the first term in the above as $\frac{1}{d}\langle \omega, e_1 + e_2 \rangle^2$ and then use equation (76) so that (85) simplifies nicely to

$$\delta^2(e_1 \ominus e_2)(d) = \langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \tag{86}$$

Plugging this into (77), we get:

$$\delta^2\left(\frac{d}{\psi}\right) = \frac{1}{\psi}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle - 2\frac{\delta(d)\delta(\psi)}{\psi^2} + 2\frac{d(\delta(\psi))^2}{\psi^2} \geq 0 \tag{87}$$

Now, we know that $\delta(e_1 \ominus e_2)(d) = -\langle \omega, e_1 + e_2 \rangle$ and $\delta(e_1 \ominus e_2)(l) = -2$, so we can go ahead and plug these into (87), obtaining

$$\delta^2\left(\frac{d}{\psi}\right) = \frac{1}{\psi}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle - \frac{4}{\psi^2}\langle \omega, e_1 + e_2 \rangle \cos\left(\frac{l\pi}{L}\right) + 8\frac{d}{\psi^3}\cos^2\left(\frac{l\pi}{L}\right) \geq 0 \tag{88}$$

but by simplifying the middle term with (76) we have the entire relation reducing to

$$\delta^2\left(\frac{d}{\psi}\right) = \frac{1}{\psi}\langle \omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t) \rangle \geq 0 \tag{89}$$

and so we have the same result as in Case 1.

We are now ready to consider the time derivative of d/ψ. We can start, as before, by writing

$$\left(\frac{d}{\psi}\right)_t = \frac{d_t}{\psi} - \frac{d}{\psi^2}\psi_t = \frac{1}{\psi}\langle\omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{L\psi}\int \kappa^2 ds - \frac{d}{\psi^2}\psi_t \tag{90}$$

We already have the time derivative of d from the proof of the theorem for open curves, but now we need an expression for ψ_t. We shall also need to recall the expression for l_t from the other proof.

$$\psi_t = \frac{L}{\pi}\cos\left(\frac{l\pi}{L}\right)\left(\frac{\pi}{L}\right) l_t = \cos\left(\frac{l\pi}{L}\right)\left(\frac{l}{L}\int \kappa^2 ds - \int_p^q \kappa^2 ds\right) \tag{91}$$

We're then able to rewrite (90) as

$$\begin{aligned}
\left(\frac{d}{\psi}\right)_t &= \frac{1}{\psi}\langle\omega, \vec{\kappa}(v,t) - \vec{\kappa}(u,t)\rangle + \frac{d}{L\psi}\int \kappa^2 ds - \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\left(\frac{l}{L}\int \kappa^2 ds - \int_p^q \kappa^2 ds\right) \\
&\geq \frac{d}{L\psi}\int \kappa^2 ds + \frac{d}{\psi^2}\cos\left(\frac{l\pi}{L}\right)\left(\int_p^q \kappa^2 ds - \frac{l}{L}\int \kappa^2 ds\right) \\
&= \frac{dl}{\psi^2}\left(\frac{\psi}{lL}\int \kappa^2 ds + \frac{1}{l}\cos\left(\frac{l\pi}{L}\right)\int_p^q \kappa^2 ds - \frac{1}{L}\cos\left(\frac{l\pi}{L}\right)\int \kappa^2 ds\right)
\end{aligned}$$

We are interested in saying something about the sign of this expression. So, we can ignore the $\frac{dl}{\psi^2}$, which is of course positive, and restrict our attention to the content inside the parentheses. Note that we can rewrite this portion as

$$\frac{1}{L}\left(\frac{\psi}{l} - \cos\left(\frac{l\pi}{L}\right)\right)\int \kappa^2 ds + \frac{1}{l}\cos\left(\frac{l\pi}{L}\right)\int_p^q \kappa^2 ds \tag{92}$$

Although until now we've done things in some generality with an L term, we now point out that we initially stated the theorem for curves of length 2π. Also, it is useful to return to the explicit definition of ψ. Using these two substitutions gives

$$\frac{1}{L}\left(\frac{2}{l}\sin\left(\frac{l}{2}\right) - \cos\left(\frac{l}{2}\right)\right)\int \kappa^2 ds + \frac{1}{l}\cos\left(\frac{l}{2}\right)\int_p^q \kappa^2 ds \tag{93}$$

Returning once again to our initial assumptions, we had said that $l(p,q,t_0) < L_{t_0}/2$, so $l < \pi$ and we know that the rightmost term in (93) is positive. We will now show that the leftmost term is positive as well. Let $x \equiv l/2$. Then the portion in parentheses becomes

$$\frac{1}{x}\sin(x) - \cos(x) = \cos(x)\left(\frac{1}{x}\tan(x) - 1\right) = \frac{1}{x}\cos(x)(\tan(x) - x) \tag{94}$$

For the same reason that we could say the rightmost term in (93) was positive, we can say now that $\frac{1}{x}\cos(x)$ is positive. So we are left to decide whether or not $\tan(x) - x$ is positive. That this is true between 0 and $\pi/2$ is easily verifiable. For example, simply examine the Taylor expansion for the range in which we are interested; for $|x| < \pi/2$: $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots$. Hence, we've shown that each of the terms in (93) is positive, and therefore $\left(\frac{d}{\psi}\right)_t \geq 0$, as we wanted to show. This completes the proof of the theorem. $\square$