Conjecture on Uniqueness of Critical Curves for Knot Energy

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Abstract

We provide a conjecture stating that the circle is unique as a critical curve for a version of curve energy from [3] and explore means for a proof involving curve shortening flow and some isoperimetric results from [1]. We also provide reason to suspect the incomplete portion of the proof will be true.

1 Backround

Several articles, including [3], [2], and [4] have explored similar types of knot energies. [4] proves that for a broad class of these energies, the circle is the global energy minimizer. One unanswered question is whether there exist any embedded curves in the plane which are local minimizers for knot energy other than the global minimizer of the circle. We will expore an element of the class of energies defined by [4] for the case where $j = 2$ and $p = 1$.

2 Critical Curves

Conjecture 1. The round circle is the only local minimizer for knot energy.

The proof follows the following structure. Gage demonstrates that under curvature flow, all sufficiently smooth embeded curves have an isoperimetric ratio approaching that of the round circle [1]. As the isoperimetric ratio is scale invariant, we can modify the curvature flow to include a component which rescales to maintain constant length. This flow is thus a total length preserving flow which evolves any embedded curve to a round circle. We can examine the Gâteaux derivative of our energy functional under this rescaled curvature flow for an arbitrary embedded, sufficiently smooth curve. If we can demonstrate that this is less than or equal to zero with equality if and only if our curve is a round circle, then the rescaled curvature flow will always immediately decrease the energy of the curve unless our curve is the round circle. This implies that there cannot be any other critical curves, for they would be curves for which any small evolution would increase the curve's energy, but since our flow always decreases energy, this cannot be. Thus we would be able to conclude that the circle is the only critical curve for our energy functional.

3 Definitions

Define:

$$\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to R^2$$

to be a unit speed C^2 curve of length 2π embedded in the plane. Abrams, Cantarella, and Foo's E_2^1 energy of a unit speed, C^2 curve γ is defined as [4]:

$$E_2^1[\gamma] = 2\int_0^{2\pi}\int_0^{\pi} \frac{1}{|\gamma(u+s)-\gamma(u)|^2} - \frac{1}{s^2}\,\mathrm{d}s\,\mathrm{d}u$$

As we are examining the behavior of knot energy under a variational field which does not maintain unit speed, we require that the energy we use continue to converge as the curve loses its unit speed parameterization. As E_2^1 defined above will diverge for non unit speed curves, we define knot energy for the purpose of this paper as follows:

$$E[\gamma] = 2\int_0^{2\pi}\int_0^{\pi} \frac{1}{|\gamma(u+s)-\gamma(u)|^2} - \frac{1}{|\gamma'(u)|^2 s^2}\,\mathrm{d}s\,\mathrm{d}u$$

Note that since our curve is initially of unit speed, the curvature is equal to the magnitude of the second derivative of our curve. We will define our rescaled curvature flow by the variational field:

$$\vec{h} = \gamma''(u) + \frac{\gamma(u)\int_0^{2\pi}|\gamma''(\phi)|\,\mathrm{d}\phi}{2\pi}$$

4 Computation

The next step is to compute the Gâteaux derivative of knot energy under our rescaled curvature flow.

$$E[\gamma] = 2\int_0^{2\pi}\int_0^{\pi} \frac{1}{|\gamma(u+s)-\gamma(u)|^2} - \frac{1}{|\gamma'(u)|^2 s^2}\,\mathrm{d}s\,\mathrm{d}u \tag{1}$$

$$\vec{h} = \gamma''(u) + \frac{\gamma(u)\int_0^{2\pi}|\gamma''(\phi)|\,\mathrm{d}\phi}{2\pi}$$

Henceforth we will write $\int \kappa^2$ to mean $\int_0^{2\pi} |\gamma''(\phi)|\,\mathrm{d}\phi$.

Let $f(u, s, \gamma(u+s), \gamma(u), \gamma'(u+s), \gamma'(u))$ be the integrand of (1). We can then take the partial derivative of f with respect to each of these symbolic independent variables to get a 'gradiant' of f:

$$\nabla f = \left\langle f_u, f_s, \frac{-2(\gamma(u+s)-\gamma(u))}{|\gamma(u+s)-\gamma(u)|^4}, \frac{2(\gamma(u+s)-\gamma(u))}{|\gamma(u+s)-\gamma(u)|^4}, 0, \frac{2\gamma'(u)}{|\gamma'(u)|^4 s^2} \right\rangle$$

and our variational field $\vec{h}$ as it affects each symbolic independent variable given by:

$$\overset{\leftrightarrow}{h}= \left\langle 0, 0, \gamma''(u+s) + \frac{\gamma(u+s)\int\kappa^2}{2\pi}, \gamma''(u) + \frac{\gamma(u)\int\kappa^2}{2\pi}, \gamma'(u+s)_t, \gamma'''(u) + \frac{\gamma'(u)\int\kappa^2}{2\pi} \right\rangle$$

Our Gâteaux derivative of knot energy under this flow is given by $\frac{\partial}{\partial\epsilon}|_{\epsilon=0}\ E_2^1[\gamma+\epsilon h] - E_2^1[\gamma]$ which we can write as $2\int_0^{2\pi}\int_0^{\pi} \nabla f\cdot \overset{\leftrightarrow}{h}\ \mathrm{d}s\,\mathrm{d}u$ which given a unit speed curve equals:

$$4\int_0^{2\pi}\int_0^{\pi} \frac{-1}{|\gamma(u+s)-\gamma(u)|^2}\left[\frac{\langle\gamma(u+s)-\gamma(u), \gamma''(u+s)-\gamma''(u)\rangle}{|\gamma(u+s)-\gamma(u)|^2} + \frac{\int\kappa^2}{2\pi}\right] ds\,du \qquad (2)$$

It remains to be shown, and in fact we fail to prove that this quantity is always less than or equal to zero with equality if and only if $\gamma(u)$ is in fact the round circle. This is the one step missing from proving the conjecture. If it is true, it would mean that given any unit speed embedded curve in the plane, its evolution according to rescaled curve-shortening flow would always immediately decrease knot energy. Since any critical curve which locally minimizes knot energy would have no evolution that immediately decreases energy, this would imply that there are no such critical curves other than the circle.

It is however worthwhile to demonstrate that this expression equals zero for the unit circle, that it converges for all sufficiently smooth curves, and examine how the various terms behave to provide reason to suspect that (2) is less than or equal to zero for all curves.

5 The Circle

Let us define:

$$\gamma : S^1 = \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$$

given by:

$$\gamma(u) = \begin{pmatrix}\cos(u)\\ \sin(u)\end{pmatrix}$$

First note that for the unit circle as we have defined it here, $\gamma''(u) = -\gamma(u)$. If we make this one substitution into (2) we get the following:

$$4\int_0^{2\pi}\int_0^{\pi} \frac{-1}{|\gamma(u+s)-\gamma(u)|^2}\left[\frac{\langle\gamma(u+s)-\gamma(u), -(\gamma(u+s)-\gamma(u))\rangle}{|\gamma(u+s)-\gamma(u)|^2} + \frac{\int\kappa^2}{2\pi}\right] \mathrm{d}s\,\mathrm{d}u$$

The affected quotient simplifies to -1. Note also that as $\kappa(\phi)^2 = |\gamma''(\phi)|^2 = 1$, we have $\int\kappa^2 = 2\pi$ and the difference within brackets in 2 reduces to zero for all s and u. Thus, as expected, the rescaled curve shortening flow does not change the knot energy of the unit circle.

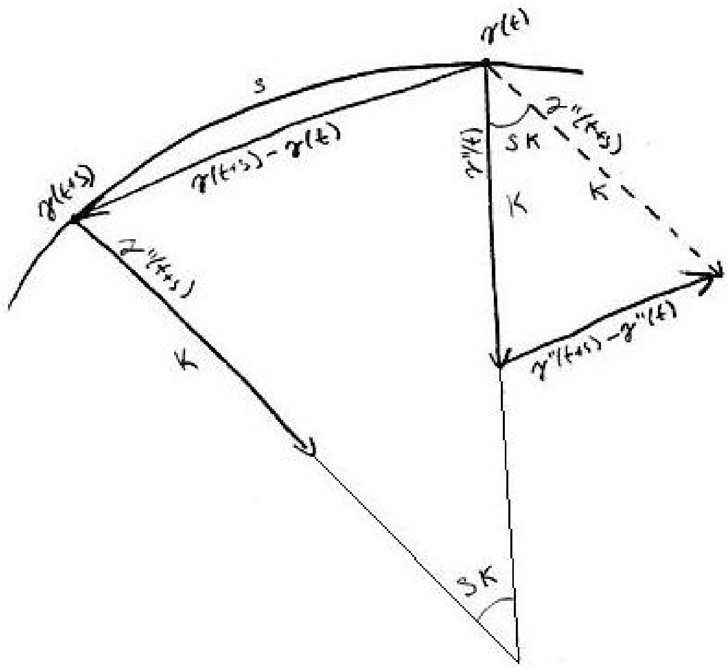

Figure 1: Supposing that two points both lie along a continuous section of circle arc allows us to greatly simplify (2)

6 Convergence

It is natural to look at (2) with some skepticism as at first glance, it would appear to be in great danger of diverging. For small s we have $|\gamma(u+s)-\gamma(u)|^2$ behaving like s^2. Thus we would seem to be in danger of our $\dfrac{-1}{|\gamma(u+s)-\gamma(u)|^2}$ term blowing up faster than the subsequent term can compensate. To evaluate this behavior, let us examine as s is integrated from 0 to some small $\epsilon > 0$. For small s we can approximate any section of $\gamma(u)$ by a circle arc with constant curvature equal to $|\gamma''(u)|$. For this calculation, let $\kappa_u = |\gamma''(u)|$. The circle arc we are approximating γ with has radius $R = 1/\kappa_u$, and so the angle subtended by the arclength s is $s\kappa_u$. See Figure. This is the angle between $\gamma''(u)$ and $\gamma''(u+s)$, each with magnitude κ_u. The difference between $\gamma''(u)$ and $\gamma''(u+s)$ is of magnitude $2\kappa_u \sin(s\kappa_u/2)$. By a similar argument, we can derive the chord between $\gamma(u)$ and $\gamma(u+s)$ to be of magnitude $(2/\kappa_u)\sin(s\kappa_u/2)$. By symmetry, both $\gamma''(u)$ and $\gamma''(u+s)$ have the same component perpendicular to the chord, and so $\gamma(u+s)-\gamma(u)$ and $\gamma''(u+s)-\gamma''(u)$ are parallel. We can see in the diagram that they point in opposite directions, however, and so

$$\langle \gamma(u+s)-\gamma(u), \gamma''(u+s)-\gamma''(u)\rangle = -4\sin^2\left(\frac{s\kappa_u}{2}\right)$$

and

$$|\gamma(u+s)-\gamma(u)|^2 = \left(\frac{4}{\kappa_u^2}\right)\sin^2\left(\frac{s\kappa_u}{2}\right)$$

substituting into (2) and simplifying we have

$$4\int_0^{2\pi}\int_0^{\epsilon}\frac{-\kappa_u^2}{4\sin^2(s\kappa_u/2)}\left[-\kappa_u^2+\frac{\int\kappa^2}{2\pi}\right]\mathrm{d}s\,\mathrm{d}u \tag{3}$$

A first order approximation of $\sin(x)\approx x$ yields

$$4\int_0^{2\pi}\int_0^{\epsilon}\frac{-1}{s^2}\left[-\kappa_u^2+\frac{\int\kappa^2}{2\pi}\right]\mathrm{d}s\,\mathrm{d}u$$

which separates into

$$4\left(\int_0^{2\pi}-\kappa_u^2\,\mathrm{d}u+\frac{\int\kappa^2}{2\pi}\int_0^{2\pi}\mathrm{d}u\right)\int_0^{\epsilon}\frac{-1}{s^2}\,\mathrm{d}s \tag{4}$$

From (4) we can see that for small s, the integral evaluates to zero, and thus our first variation converges as desired.

7 Boundedness

It will remain to be proven that (2) is indeed less than or equal to zero for all appropriate curves with equality if and only if our curve is the unit circle. However examining certain local behavior, we can come to a very strong suspicion that this inequality will in fact hold. Unfortunately, an attempt to satisfy the inequality pointwise will fail, as the following circumstance will demonstrate. First note that the approximations we made to arrive at (3) assumed only that $\gamma(u)$ and $\gamma(u+s)$ both lie on the same circle arc. Also note that if the term in brackets is always greater than zero, then the integrand evaluates to be less than zero as desired. $\displaystyle\int\kappa^2$ is a constant for the curve and is at least equal to one. If we have a curve with a section of circle arc with $|\gamma''(u)|>1$ then we could have a point where the integrand evaluates to be greater than zero. However, a curve having a section with $|\gamma''(u)|>1$ will also inflate $\displaystyle\int\kappa^2$, so it is also possible that this is not a problem. Unfortunately, the curves we care about are not constant curvature curves, and thus the integral will not simplify as nicely for all possibilities.

7.1 Bending Energy

Let us examine $\displaystyle\int\kappa^2$, which has also been referred to as bending energy, to better understand its behavior for different curves. For our curve γ let us explore the arclength parameterized signed curvature function

$$\kappa:\mathbb{R}/2\pi\mathbb{Z}\to\mathbb{R}$$

As Exner reminds us, $\kappa(t)$ provides enough information to define our curve γ up to Euclidean transformations, translation and rotation. Thus it is sufficient to work with $\kappa(t)$ as (2) is independent of Euclidean transformations. For any closed, embedded, C^2 curve, we will have $\int_0^{2\pi} \kappa(t)\,\mathrm{d}t = 2\pi$. Instead of working directly with $\kappa(t)$, let us work with $\kappa_0(t)$ which we define to be equal to $\kappa(t) - 1$. Notice that $\int_0^{2\pi} \kappa_0(t)\,\mathrm{d}t = 0$. With an appropriate substitution, we find that $\int_0^{2\pi} \kappa(t)^2\,\mathrm{d}t = 2\pi + \int_0^{2\pi} \kappa_0(t)^2\,\mathrm{d}t$, thus proving the earlier assertion that $\int \kappa^2 \geq 2\pi$, and demonstrating that there is equality if and only if our curve is the circle, the only curve with $\kappa_0(t) = 0$.

7.2 A pointwise argument

Let us concentrate on the term of (2) in brackets:

$$\frac{\langle \gamma(u+s) - \gamma(u), \gamma''(u+s) - \gamma''(u) \rangle}{|\gamma(u+s) - \gamma(u)|^2} + \frac{\int \kappa^2}{2\pi} \tag{5}$$

If this is greater than or equal to zero for all $t \in [0, 2\pi]$ and $s \in [0, \pi]$ then (2) will evaluate to be less than or equal to zero for all appropriate curves, as desired.

Recall that for s, t on a circle arc, we showed in a previous section that

$$\frac{\langle \gamma(u+s) - \gamma(u), \gamma''(u+s) - \gamma''(u) \rangle}{|\gamma(u+s) - \gamma(u)|^2} = -\kappa(t)^2$$

Suppose we have two points on our curve, $\gamma(t)$ and $\gamma(t+s)$ such that they both lie along a continuous circle arc of curvature greater than one. We then must examine the least possible amount by which $\int \kappa^2$ must increase in order to see if we can build a pointwise violation, or show that one does not present itselt. From t to $t+s$ we have κ_0 equals some constant, call it κ_1. In order to minimize $\int \kappa^2$ restricted to $\int_0^{2\pi} \kappa_0(t)\,\mathrm{d}t = 0$ then κ_0 must be equal to $\frac{-s\kappa_1}{2\pi - s}$ elsewhere. As a result, (5) evaluates to

$$s\kappa_1^2 + \frac{s^2\kappa_1^2}{2\pi - s} - 2\pi\kappa_1^2 - 4\pi\kappa_1$$

Unfortunately, we see that plugging in $s = \pi$ yields $(1-\pi)\kappa_1^2 - 4\pi\kappa_1$, which given $\kappa_1 > 0$ yields a negative result. This however, does not doom our result. While the class of curves we constructed did satisfy some of the restrictions on the types of curves we care about, they actually are not closed, and thus the only result we can take from this approximation is that the approximation is not good enough. It is likely that the restrictions Exner places on the variation of his curvature function will turn out to be exactly the restriction we would like to place on our $\kappa_0(t)$ function in order to be sure we are defining an appropriate curve. Using that restriction and thus a Fourier series to define $\kappa_0(t)$ it is possible that (5) will be positive for all appropriate s and t, bounding (2) pointwise as we would like. Alternatively, (2) might not be boundable in a pointwise fashion, yet still hold in which case some other technique will be needed to evaluate the expression over some arbitrary curve.

References

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