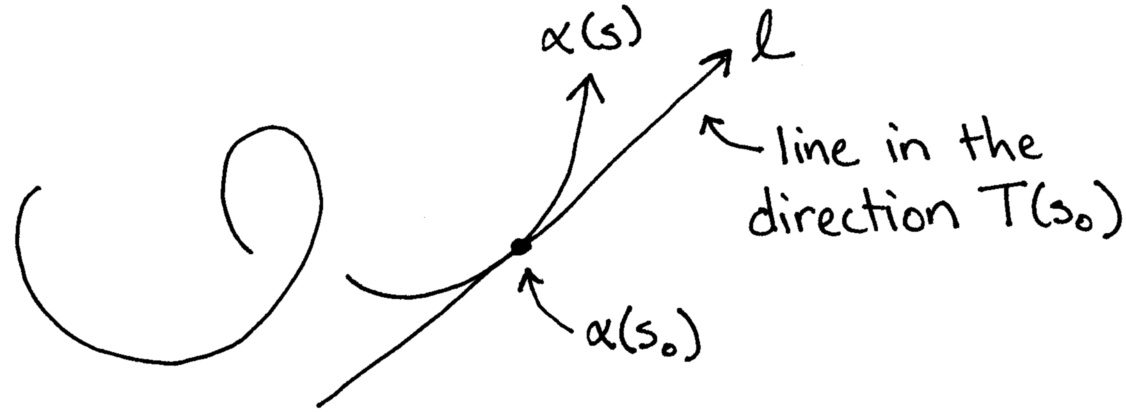
The Tangent Plane and Differential

We now want to consider the tangent plane to a surface parametrized by

$$X : U \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3.$$

We consider, by analogy, the tangent line to a curve:



The line through $\vec{p}$ in direction $\vec{v}$ has the equation

$$\ell(t) = \vec{p} + t\vec{v}$$

so the tangent line to $\alpha(s)$ at s_0 is given by

$$\begin{aligned} \ell(t) &= \alpha(s_0) + t\,T(s_0) \\ &= \alpha(s_0) + t\,\alpha'(s_0). \end{aligned}$$

It is helpful to think of $\alpha'(s_0)$ as the differential of the map

$$\alpha : U \subset \mathbb{R} \to \mathbb{R}^3,$$

or as

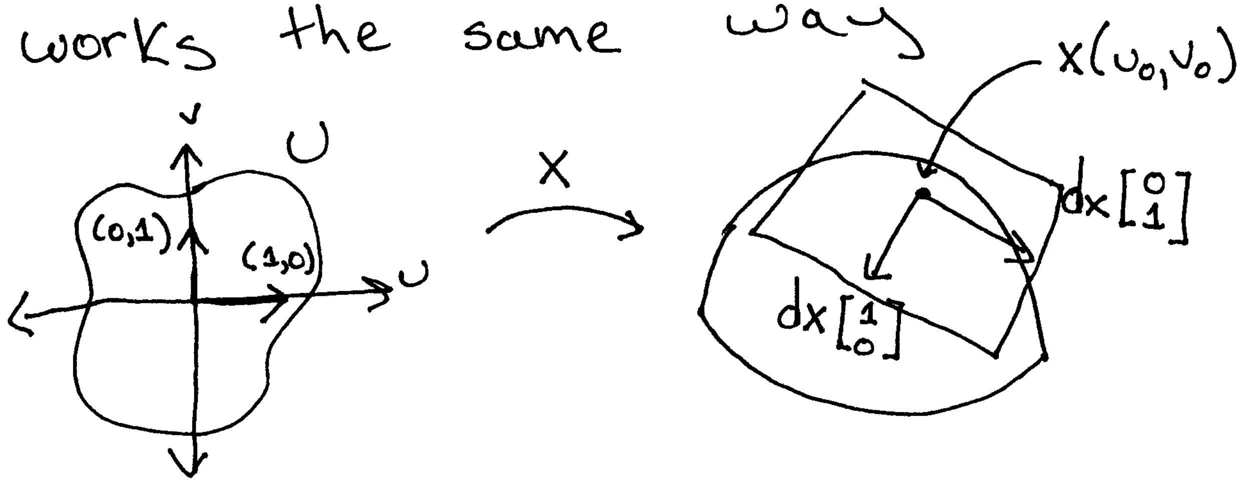
$$d\alpha : \mathbb{R} \to \mathbb{R}^3, \quad d\alpha = \begin{bmatrix} \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial s} \end{bmatrix}$$

We can then think of the tangent line ℓ as

$$\alpha(s_0) + \text{Image } d\alpha$$

~~The same understanding~~

For a surface, the tangent plane works the same way



The tangent plane is the span of

$$\vec{X}_u = \begin{bmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{bmatrix} \quad \text{and} \quad \vec{X}_v = \begin{bmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{bmatrix}$$

~~We~~ ~~with~~ translated by $\vec{X}(u,v)$. We recall that the equation of the plane normal to $\vec{n}$ through $\vec{p}$ is

$$\vec{n} \cdot (x,y,z) = \vec{n} \cdot \vec{p}.$$

Since this plane is normal to $\vec{X}_u \times \vec{X}_v = \vec{n}$ we can use this formula to explicitly compute an example.

Example. Consider the sphere

$$\vec{X}(\theta,\varphi) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi).$$

Find the tangent plane at $(\pi/3, \pi/4)$.

We compute

$$\vec{X}_\theta = \begin{bmatrix} -\sin\varphi\sin\theta \\ \sin\varphi\cos\theta \\ 0 \end{bmatrix} \times \begin{bmatrix} \cos\varphi\cos\theta \\ \cos\varphi\sin\theta \\ -\sin\varphi \end{bmatrix} = \vec{X}_\varphi$$

$$= \begin{bmatrix} -\sin^2\varphi\cos\theta \\ -\sin^2\varphi\sin\theta \\ -\sin\varphi\cos\varphi\sin^2\theta - \sin\varphi\cos\varphi\cos^2\theta \end{bmatrix} = -\sin\varphi \underbrace{\begin{bmatrix} \sin\varphi\cos\theta \\ \sin\varphi\sin\theta \\ \cos\varphi \end{bmatrix}}_{= \vec{X}(\theta,\varphi)}$$

So at $\theta = \pi/3$, $\varphi = \pi/4$, we have

$\sin\theta = \sqrt{3}/2$ $\qquad$ $\sin\varphi = 1/\sqrt{2}$

$\cos\theta = 1/2$ $\qquad$ $\cos\varphi = 1/\sqrt{2}$

At this point the normal vector is

$$-\frac{1}{\sqrt{2}} \begin{bmatrix} \cancel{\sqrt{3}/2\sqrt{2}}\ 1/2\sqrt{2} \\ \sqrt{3}/2\sqrt{2} \\ 1/2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/4 \\ -\sqrt{3}/4 \\ -1/4 \end{bmatrix}$$

So the tangent plane is

$$-\tfrac{1}{4}x - \tfrac{\sqrt{3}}{4}y - \tfrac{1}{4}z = -\tfrac{1}{\sqrt{2}},$$

using the fact that

$$\vec{n} = -\sin\varphi \ \vec{X}(\theta,\varphi)$$

that we noted above.

We let $\vec{n} = \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$ be the unit normal to S.
We denote the tangent plane to a surface S at p by T_pS. We note that if we have a differentiable map

$$f: S_1 \to S_2$$

between surfaces, there is a corresponding linear map

$$df_p: T_pS_1 \to T_{f(p)}S_2$$

To define this map explicitly, we need to introduce bases for the linear spaces $T_p S_1$ and $T_{f(p)} S_2$. If S_1 is parametrized by u, v coordinates then

$$T_p S_1 = \text{span}\langle X_u, X_v \rangle$$

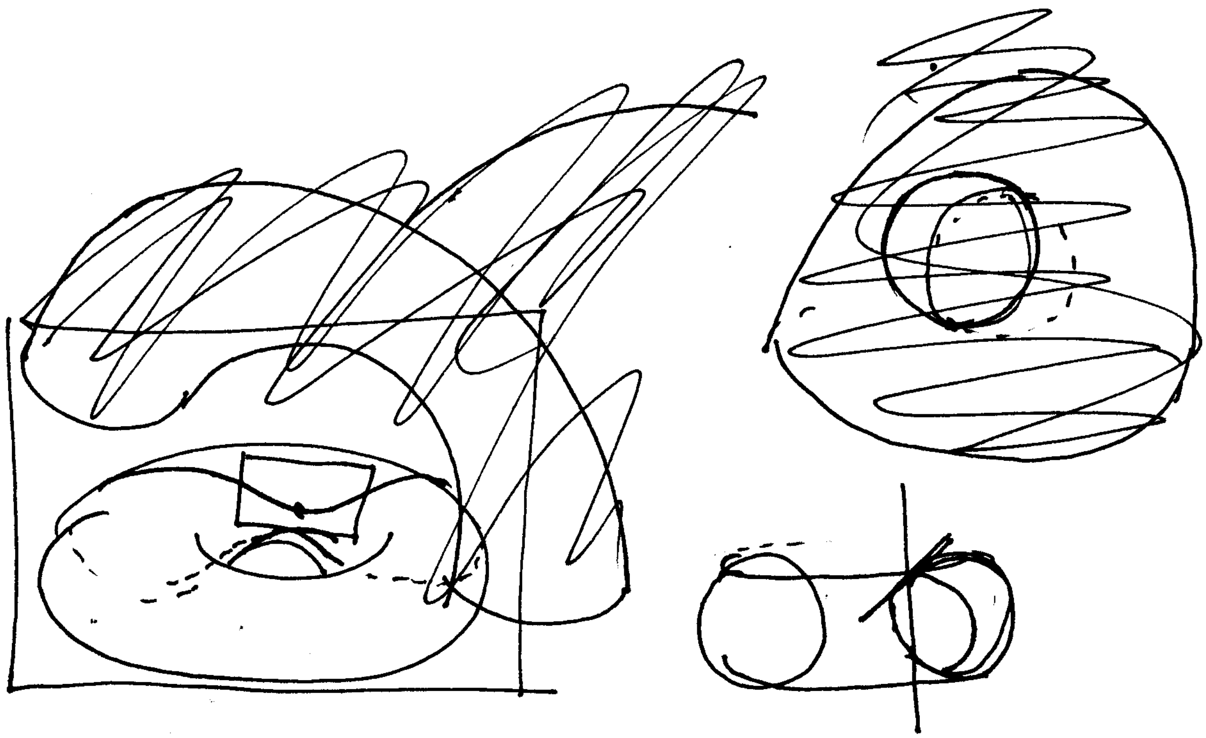
so these are the natural coordinates on the tangent plane:

$$\vec{w} \in T_p S_1 = w_1 \vec{X}_u + w_2 \vec{X}_v + \vec{p}$$

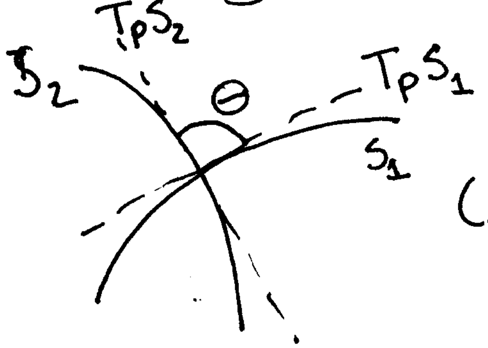
for some (w_1, w_2). If we write f in terms of local coordinates (u_1, v_1) and (u_2, v_2) on S_2, then in the w_1, w_2 coordinate system on $T_p S_1$ and $T_{f(p)} S_2$, we have $f(u_1, v_1) = (u_2(u_1, v_1), v_2(u_1, v_1))$

$$df_p = \begin{bmatrix} \frac{\partial u_2}{\partial u_1} & \frac{\partial v_2}{\partial u_1} \\ \frac{\partial u_2}{\partial v_1} & \frac{\partial v_2}{\partial v_1} \end{bmatrix}$$

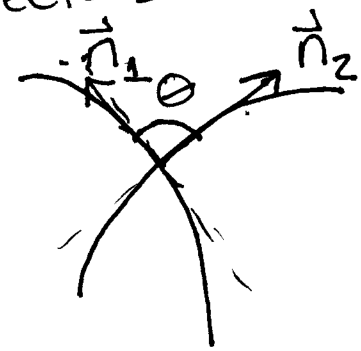
(It can be shown that this expression of df_p doesn't depend on our choice of local coordinates.)



When two surfaces intersect, the angle between them is the angle between their tangent planes, or between their normal vectors

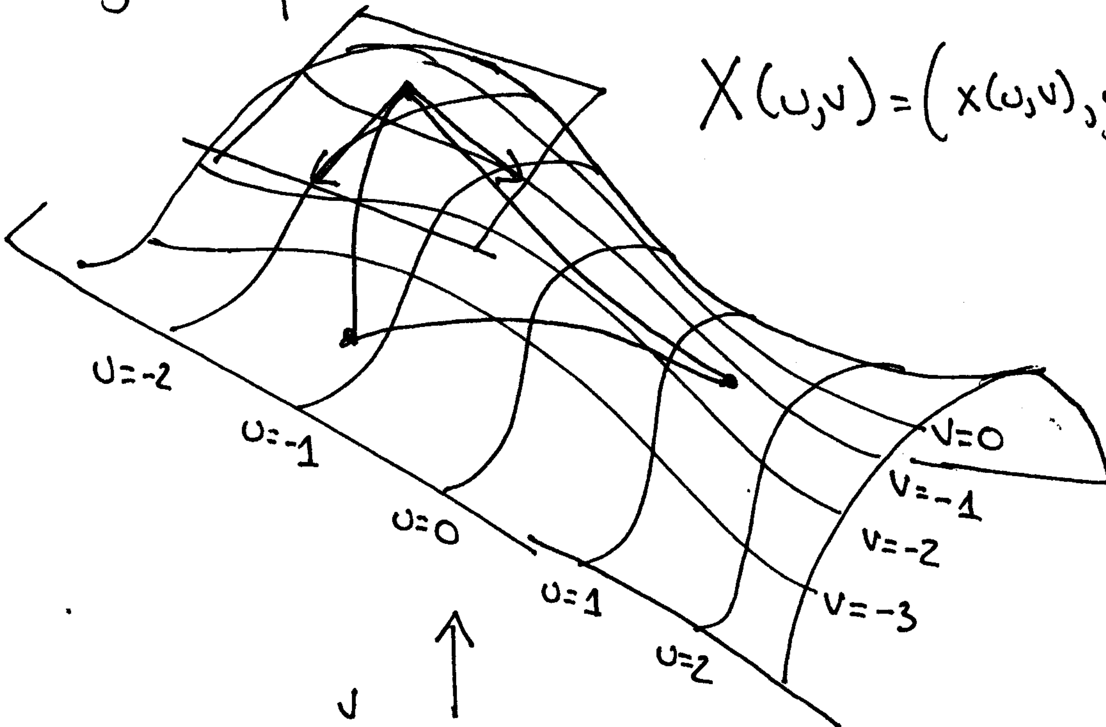


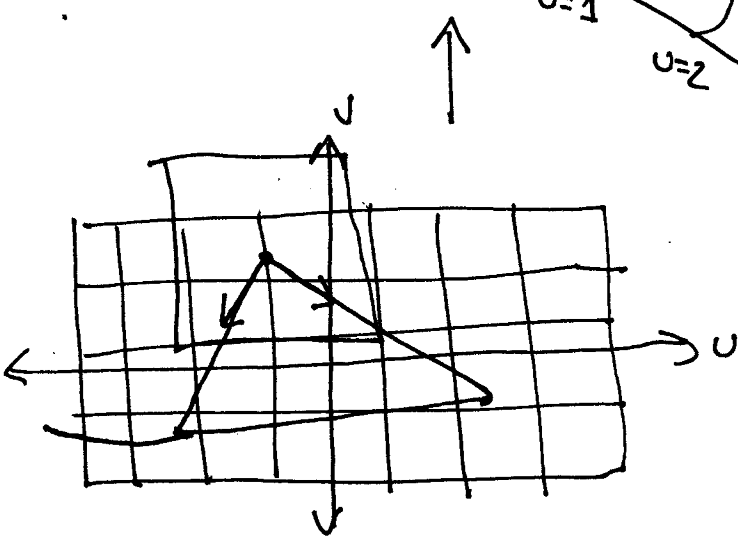
(side view)



We now can define surfaces and their tangent planes and establish local coordinates

$$X(u,v) = \big(x(u,v), y(u,v), z(u,v)\big)$$





How can we do geometry in the u,v plane? We must measure lengths, angles, and areas with respect to the geometry of the curved ~~image~~ ~~Z~~ image S of the plane!

Brief Review (or Introduction to) Quadratic Forms

We first recall some linear algebra.

$$\langle \vec{v}, \vec{w} \rangle = \vec{v} \cdot \vec{w}, \text{ for } \vec{v}, \vec{w} \in \mathbb{R}^n$$

If we have an $n \times n$ matrix A, it is not hard to see that

$$\langle A\vec{v}, \vec{w} \rangle = \langle \vec{v}, A^T \vec{w} \rangle$$

where A^T is the transpose of A. If A is a symmetric matrix, then $A = A^T$. In this case, we can define a "quadratic form" associated to A:

$$Q_A(\vec{v}, \vec{w}) = \langle \vec{v}, A\vec{w} \rangle$$

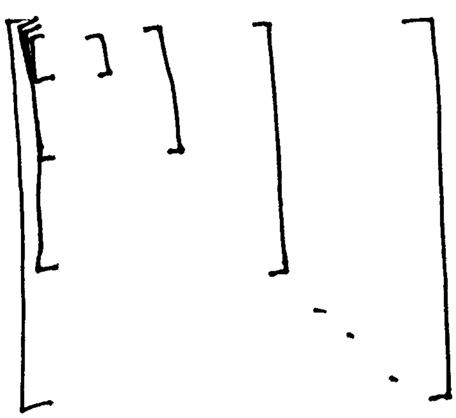
This form is a function on pairs of vectors which is

1) symmetric $Q_A(\vec{v},\vec{w}) = Q_A(\vec{w},\vec{v})$

2) bilinear $Q_A(a\vec{v}_1 + b\vec{v}_2, \vec{w}) = aQ_A(v_1, w)$
$+ b\,Q_A(v_2, w)$

~~If A has positive determinant, then~~

If all of the matrices



have positive determinant

then Q_A is

3) positive-definite

$$Q_A(\vec{v},\vec{v}) \geq 0$$

and $Q_A(\vec{v},\vec{v}) = 0 \iff \vec{v} = 0$.