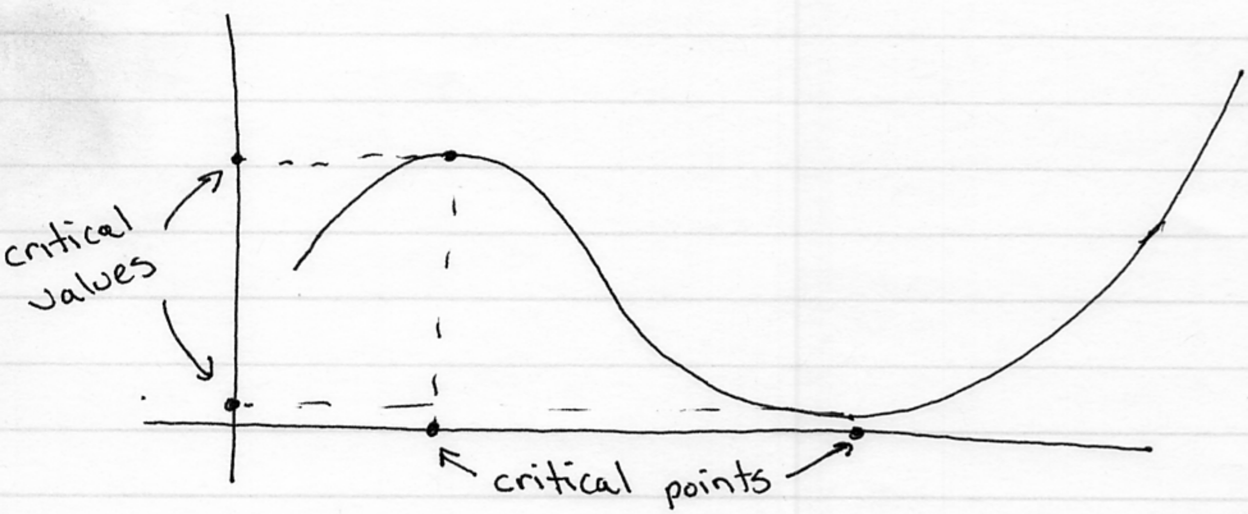
We next consider surfaces defined implicitly by functions.

Definition. Given a differentiable map $F: U \subset \mathbb{R}^n \to \mathbb{R}^m$ we say that $p \in U$ is a critical <u>point</u> of F if $dF_p : \mathbb{R}^n \to \mathbb{R}^m$ is not onto.

The image $F(p)$ is called a critical <u>value</u>. A point of $\mathbb{R}^m$ which is not a critical value is a <u>regular</u> value.



Proposition. If $f: U \subset \mathbb{R}^3 \to \mathbb{R}$ is a differentiable function and $a \in f(U)$ is a regular value, then $f^{-1}(a)$ is a regular surface in $\mathbb{R}^3$.

We won't do the proof in class (read 2-2 in Do Carmo!) but it is similar to the proof we just completed, except that it is the <u>implicit</u> function theorem which saves the day.

Examples.

The ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is a regular surface. To check, we write it as $f^{-1}(0)$, where

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1.$$

Then we find the critical values of f by computing

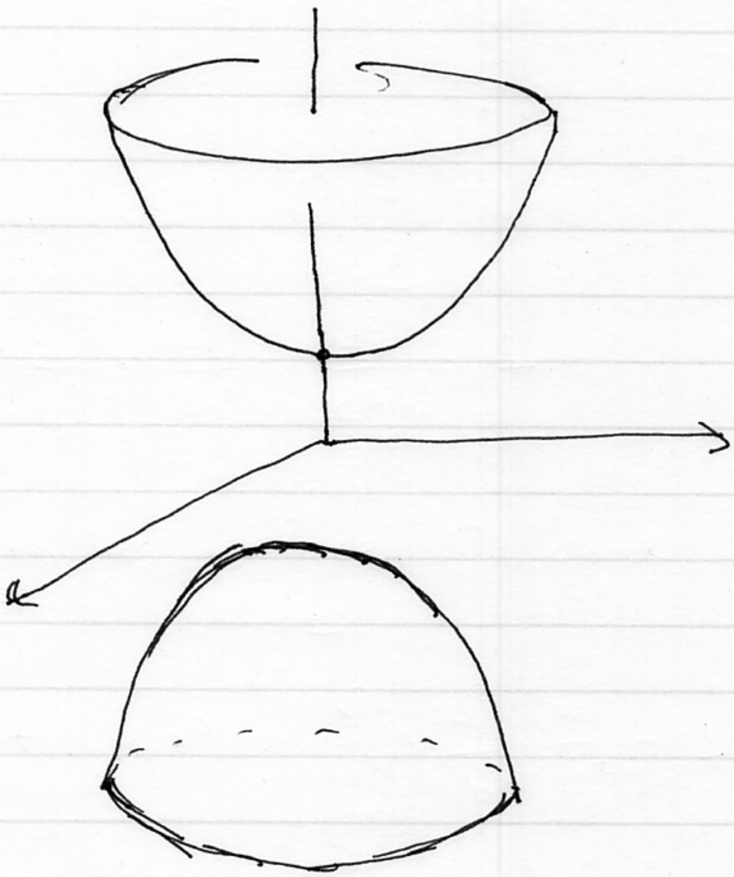
$$Df = \left[\frac{2x}{a^2} \quad \frac{2y}{b^2} \quad \frac{2z}{c^2} \right].$$

This is not onto only when x, y, z all vanish. At that point, $f(0,0,0) = 1$ so 1 is the only critical value. Hence 0 is a regular value and we're done!

Similarly the hyperboloid of two sheets

$$-x^2 - y^2 + z^2 = 1$$

is a regular surface



Notice that this surface is not connected.
(Which is fine, from our definitions.)

We now turn our attention to functions on surfaces. We would like to develop a precise understanding of what it means for a function on a surface S to be "differentiable".

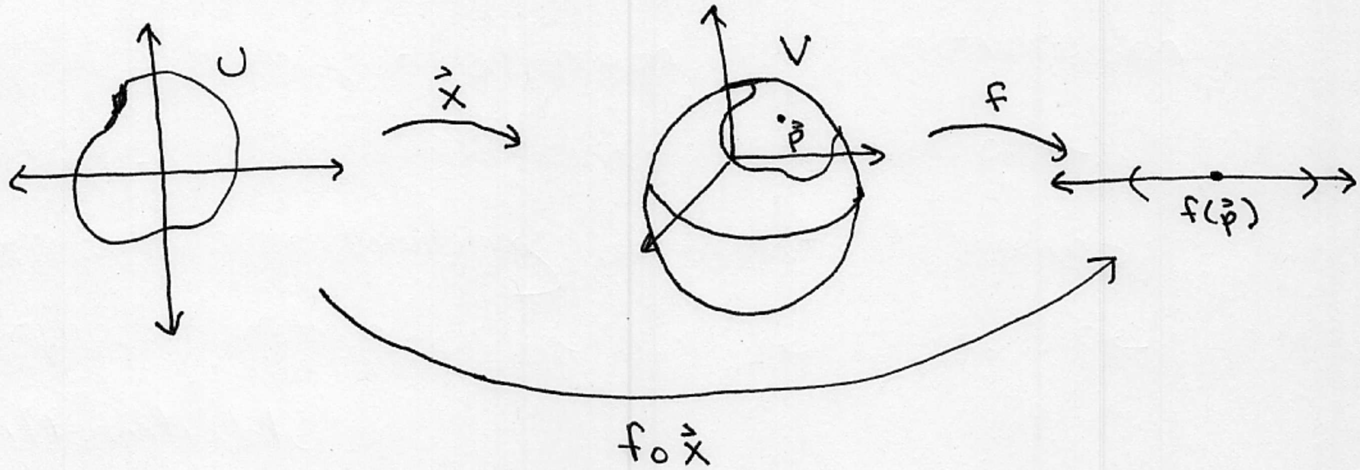
The key question is:

Differentiable with respect to <u>what</u>?

An obvious answer is provided by the definition below: with respect to local coordinates on S.

Definition. Let $f: V \subset S \to \mathbb{R}$ be a function defined in an open subset of a regular surface S. Then f is said to be <u>differentiable</u> at $\vec{p} \in S$ if for some parametrization $\vec{x}: U \subset \mathbb{R}^2 \to S$ with $\vec{x}(\vec{0}) = \vec{p}$, the composition $f \circ \vec{x}$ is differentiable at $\vec{0}$.

Here's the picture:



We are claiming (well, insisting) that f is differentiable at $\vec{p}$ on the surface iff $f \circ \vec{x}$ is differentiable as an ordinary scalar function on $\mathbb{R}^2$.

This leaves us with a problem:

> What if we picked the wrong parametrization $\vec{x}$? Could $f \circ \vec{x}$, ~~and~~ but not $f \circ \vec{y}$ be differentiable for different parametrizations $\vec{x}$ and $\vec{y}$?

Basically, no. This never happens.

Proposition (Change of Parameters Theorem)

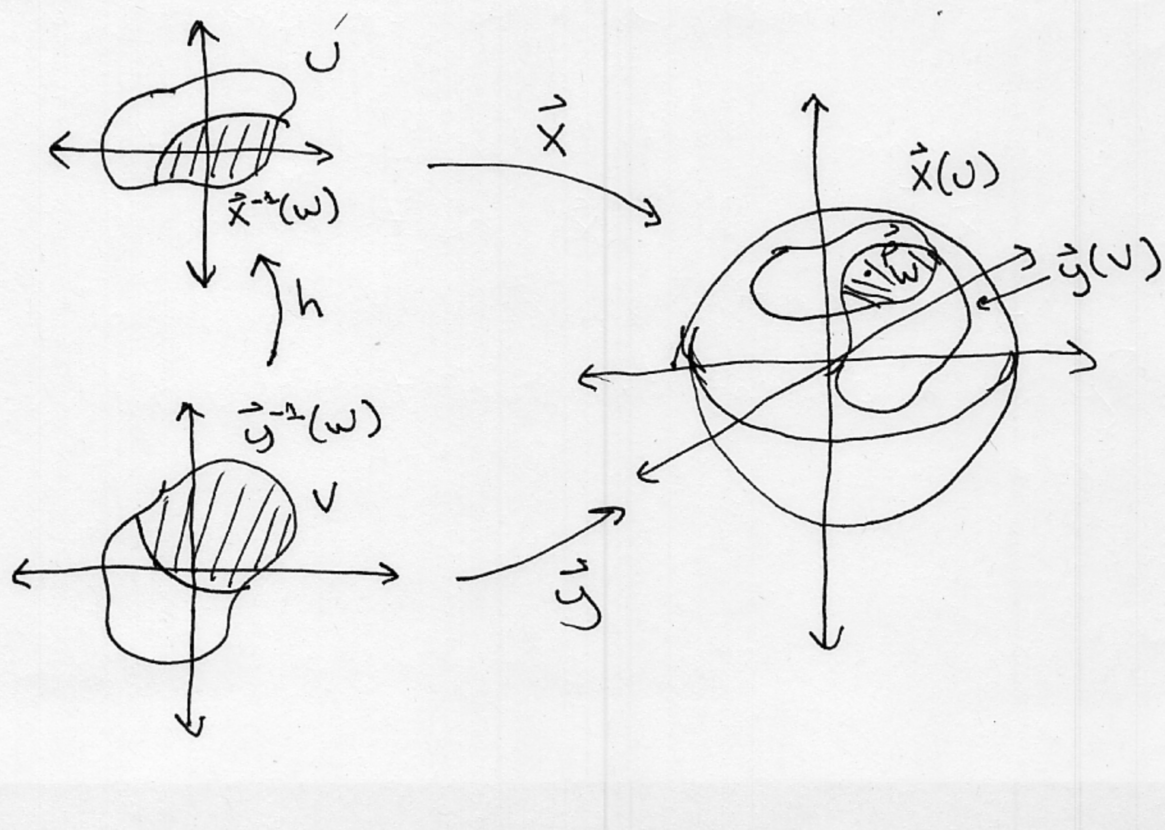
Let $\vec{p}$ be a point of a regular surface S and let $\vec{x}: U \to S$ and $\vec{y}: V \to S$ be two parametrizations of S with $\vec{p} \in \vec{x}(U) \cap \vec{y}(V) = W$.

Then the "change of coordinates" map

$$h = \vec{x}^{-1} \circ \vec{y} : \vec{y}^{-1}(W) \to \vec{x}^{-1}(W)$$

is a diffeomorphism (is differentiable <u>and</u> has a differentiable inverse).

Here's the picture



Thus any set of local coordinates u, v on S can be written as differentiable functions $u(r,s)$, $v(r,s)$ of any other set of local coordinates r, s on S.

So if $f(u,v)$ is differentiable w.r.t. u and v, it is certainly differentiable w.r.t. r and s by the chain rule:

$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial r} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial r}.$$

Example:

$S = S^2$, and f is given in the local coordinates

~~$x,y \mapsto (x, y, \sqrt{1-x^2-y^2})$ as $f(x,y) = x^2$.~~

(y,z)

Height function, square of distance.