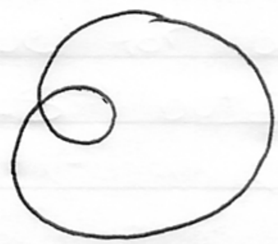
Stuff turning inside out!

The Whitney-Gravenstein Theorem.

Let's go back to the rotation index for a moment. We have a map from closed curves to S^1 given by the tangent indicatrix $\vec{t}(s)$. The number of times $\vec{t}(s)$ wraps around that circle is called the <u>rotation index</u> of α.

We notice that I counts "loops" in α:

$I = \pm 1$



$I = \pm 2$

To what extent are these loops stable features of α? Can we classify all curves by counting their "loop number"?

We first need a way to define equivalence of curves: we recall that

Definition. A curve $\alpha(t)$ is regular if $\alpha'(t) \neq 0$.

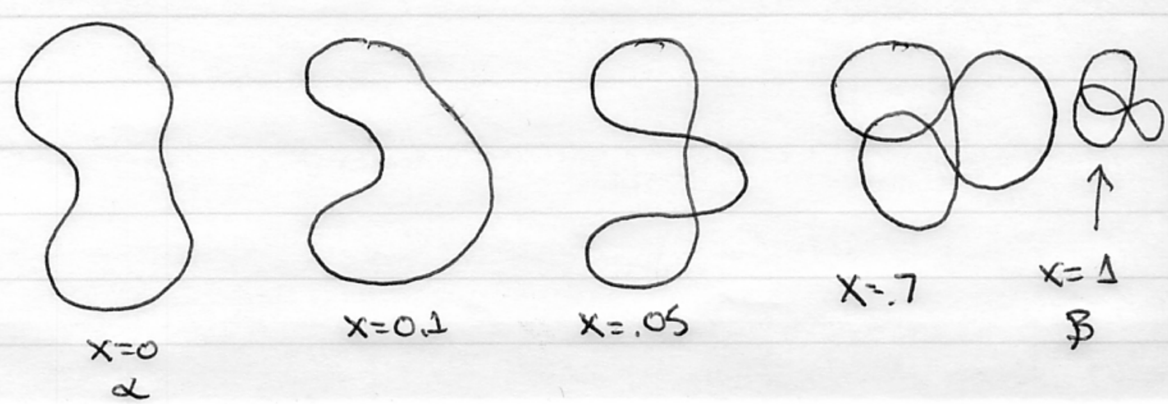
We then have:

Definition. A regular homotopy between two regular curves $\alpha(t)$ and $\beta(t)$ is a C^1 closed ~~continuous~~ map

$$H(t,x) : [a,b] \times [0,1] \to \mathbb{R}^2$$

so that $H(t,0) = \alpha(t)$, $H(t,1) = \beta(t)$ and $H(t,x_0)$ is a regular closed parametrized differentiable curve as a function of t.

We can think of H as a movie showing α sliding over to β.



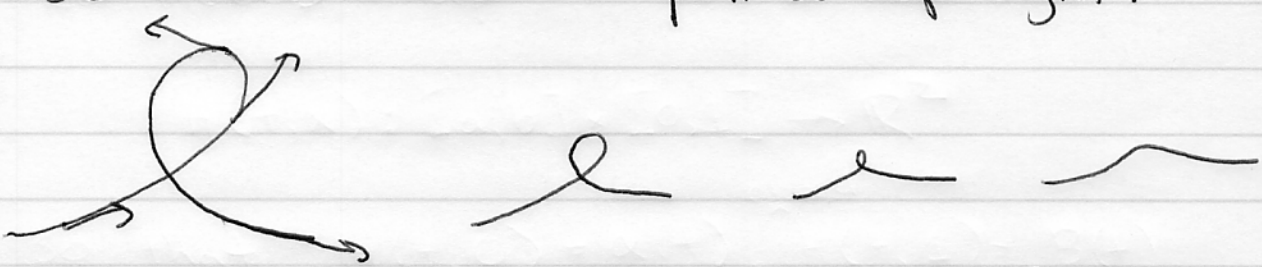
Notice that some moves are forbidden by our definition of regular homotopy.

Since H is C^1 (as a function of x and t),

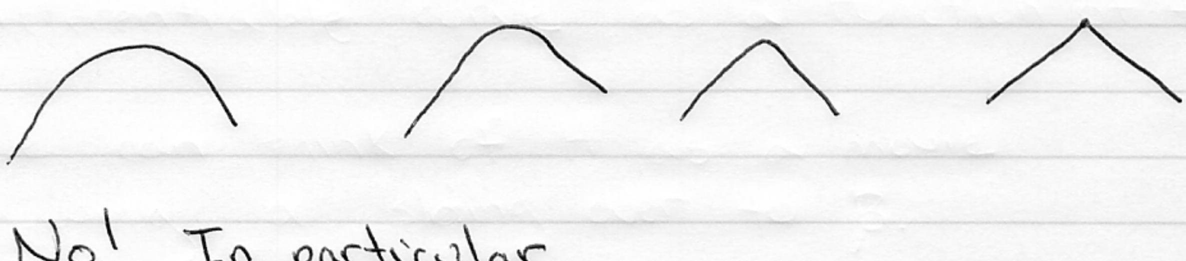
$$\frac{\partial H}{\partial t}(x,t) \text{ is } \underline{\text{continuous}}\ \underline{\text{in}}\ \underline{x}$$

So the ~~velocity~~ tangent vectors of the intermediate curves $H(t, x)$ depend continuously on x.

So can we.... pull a loop tight?



No! Can we make a corner?



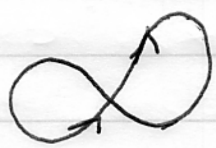
No! In particular,

Proposition. The rotation index of $H(t,x)$ depends continuously on x. Thus it's constant.

Proof. The rotation index is defined as the total angle swept out by the tangent indicatrix of $H(t, x)$.

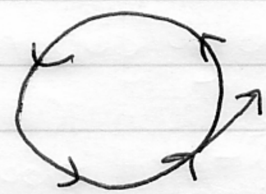
Since the tangent indicatrix depends continuously on x, this angle does too.

We now know: You can't turn a circle inside out!



$I = 0$

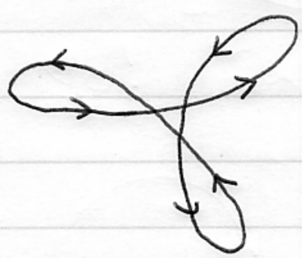
is not regularly homotopic to



$I = +1$

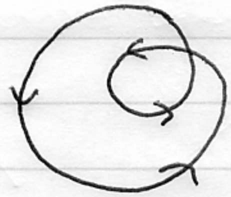
$I = -1$

Is



$I = +2$

reg. homotopic to



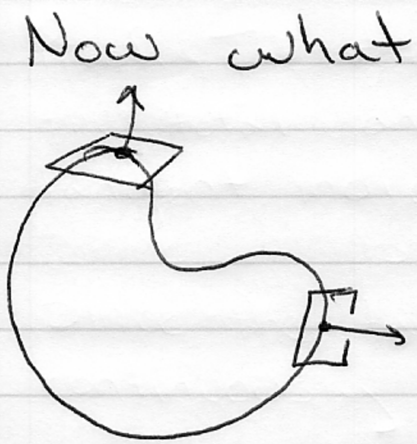
$I = +2$

?

Yes! Are all curves with the same rotation index regularly homotopic?

This is the Whitney-Gravenstein theorem!

Now what about surfaces?



At every point on a surface, there is a normal vector $\vec{n}(x)$.

These normal vectors define a Gauss map from S to the unit sphere

$$g : S \to S^2$$

If S is a (funny shaped) sphere, this is (topologically) a map

$$g : S^2 \to S^2$$

with a "rotation index" or *degree* measuring how many times g wraps around S^2.

Now if we turn the sphere inside out, we compose

$$S^2 \xrightarrow[\vec{x} \mapsto -\vec{x}]{} S^2 \xrightarrow[g]{} S^2 \xrightarrow[\vec{x} \to -\vec{x}]{} S^2$$

position vectors reversed

normal vectors reversed, too!

It turns out that degree multiplies when you compose maps, and the degree of the antipodal map is -1.

So the inside-out sphere has the same "rotation index" as the original sphere!

Is there a "Whitney-Gravenstein Theorem for surfaces" which would let us conclude these spheres are regularly homotopic?!?!

Let's find out...