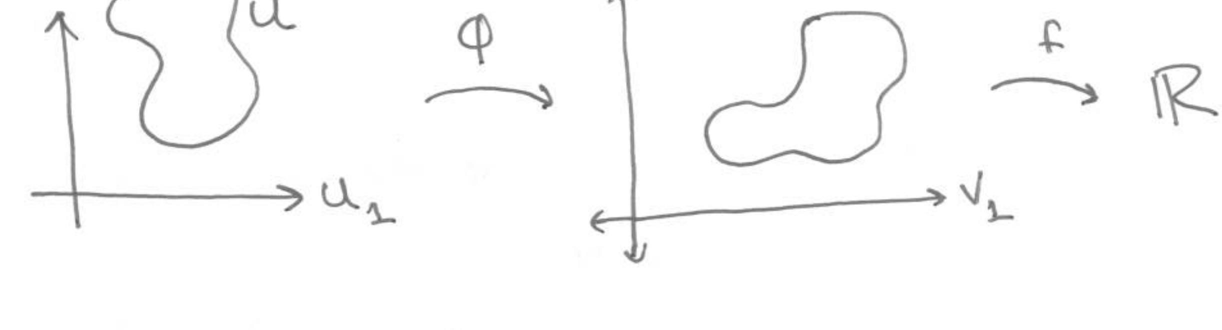
Integral Geometry II.

We now want to give another type of integralgeometric formula. Since everything will depend on the change of variables formula from multivariable calc, let's begin with a review.

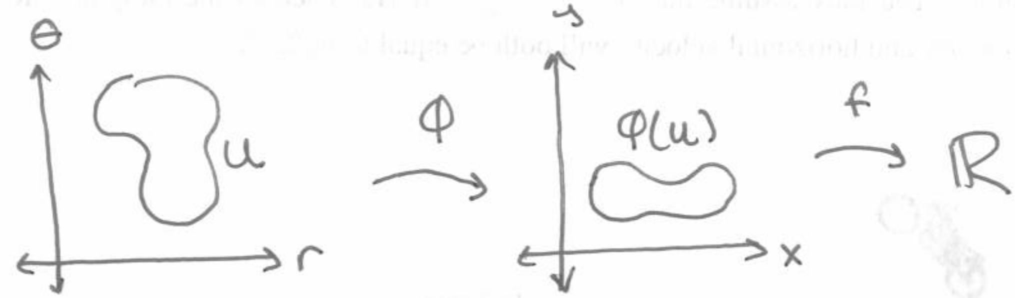
Suppose $\varphi : U^{\subset \mathbb{R}^n} \to \mathbb{R}^n$ is 1-1, and has cts. differentiable partials with $|D\varphi| \neq 0$. Then for any continuous f,

$$\int_{\varphi(U)} f(v)\, dv = \int_U f(\varphi(u))\, |D\varphi|\, du.$$



Example. Let $\varphi(r,\theta) = (r\cos\theta, r\sin\theta)$
$= (x,y)$

be the map from polar to rectangular coordinates. Then



$$\int_{\varphi(u)} f(x,y)\,dx\,dy = \int_{u} f(r,\theta)\,|D\varphi|\,dr\,d\theta$$

So what is $|D\varphi|$? This is the determinant of the matrix

$$\begin{pmatrix} \frac{\partial \varphi_1}{\partial u_1} & \frac{\partial \varphi_1}{\partial u_2} & \cdots & \frac{\partial \varphi_1}{\partial u_n} \\ \\ \frac{\partial \varphi_n}{\partial u_1} & \cdots & \cdots & \frac{\partial \varphi_n}{\partial u_n} \end{pmatrix} = \text{the ``Jacobian of } \varphi\text{''}.$$

and in our case, we have

$$\varphi_1(r,\theta) = r\cos\theta$$
$$\varphi_2(r,\theta) = r\sin\theta$$

so the determinant is

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus we have

$$\iint f(x,y)\,dx\,dy = \iint f(r,\theta)\,r\,dr\,d\theta,$$

as expected!

Now let's move on to another example.

We call a map $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ a <u>rigid motion</u> if it is a combination of rotation and translation. A rotation by θ is the linear map

$$R_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta)$$

and a translation by (v_1, v_2) is the map

$$T_{(v_1,v_2)}(x,y) = (x+v_1,\ y+v_2).$$

Assembled, these maps look like

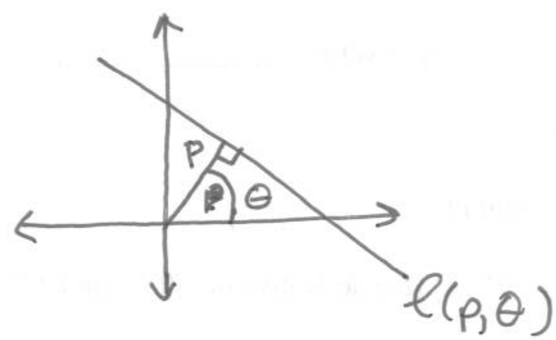
$$\varphi_{\theta,(v_1,v_2)}(x,y) = (x\cos\theta - y\sin\theta + v_1,\ x\sin\theta + y\cos\theta + v_2).$$

Now we are ready to do some geometry.

Definition. Let L be the set of lines in $\mathbb{R}^2$. We parametrize this space by setting $\ell(\theta, p)$ to be the line

$$(\cos\theta)x + (\sin\theta)y = p$$

or



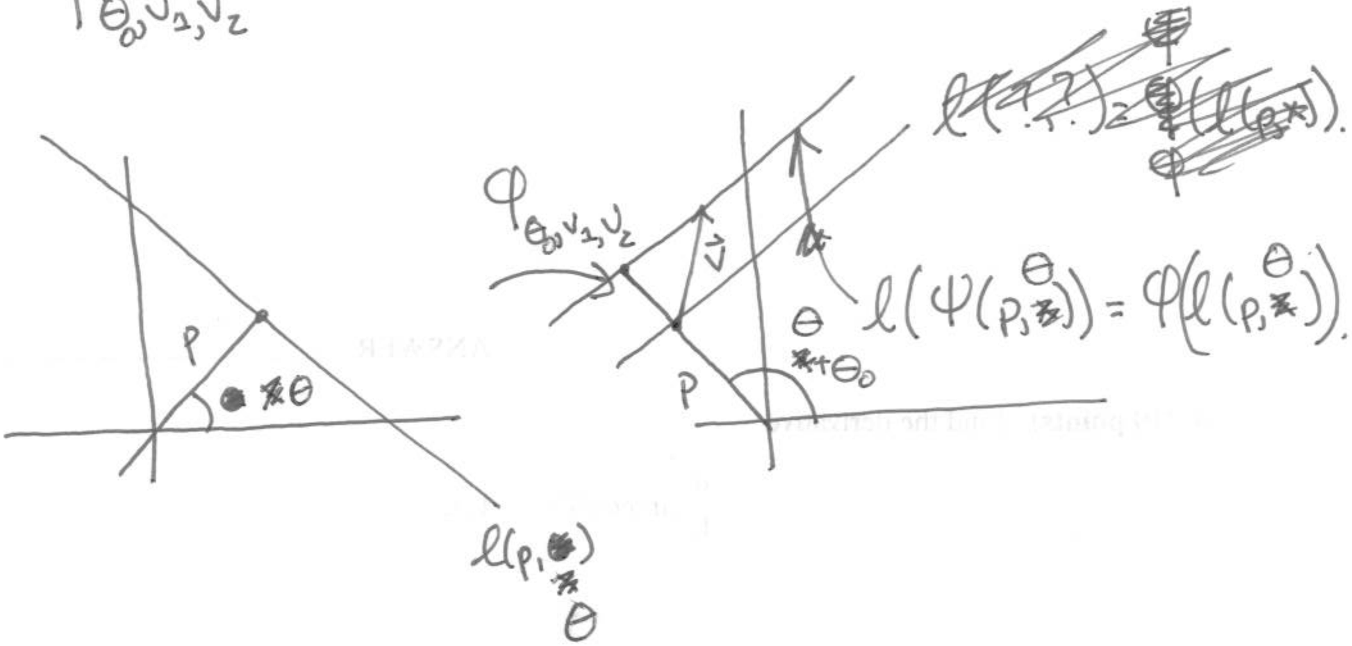
Theorem. If α is a curve in $\mathbb{R}^2$,

$$\text{Length}(\alpha) = 4 \iint I(p,\theta)\, dp\, d\theta$$

where $I(p,\theta) = \#$ of intersections of α with the line $\ell(p,\theta)$.

Here's the big picture. Suppose we have some rigid motion $\varphi_{\theta_0, v_1, v_2}$ of the plane. This map induces a map from L to L.

$\psi_{\theta_0, v_1, v_2}$



We see that

$$\psi_{\theta_0, \vec{v}}(p, \theta) = \left(p + \langle \vec{v}, (\cos(\theta+\theta_0), \sin(\theta+\theta_0)) \rangle, \theta_0 + \theta\right)$$

We now see the effect of ψ on the integral.

Let

$$I_\alpha(p,\theta) = \#\text{ of intersections of } \alpha \text{ with } \ell(p,\theta).$$

Then

$$I_\alpha(\psi(p,\theta)) = \#\text{ of intersections of } \varphi(\alpha) \text{ with } \psi(\ell(p,\theta))$$

$$= \#\text{ of intersections of } \alpha \text{ with } \ell(p,\theta).$$

Now by change of variables,

$$\int_L I_\alpha(\psi(p,\theta))\,|d\psi|\,dp\,d\theta = \int_L I_\alpha(p,\theta)\,dp\,d\theta.$$

So let's compute

$$d\psi = \begin{bmatrix} 1 & \frac{\partial}{\partial\theta}\langle \vec{v}, (\cos(\theta+\theta_0), \sin(\theta+\theta_0))\rangle \\ 0 & 1 \end{bmatrix}$$

Thus

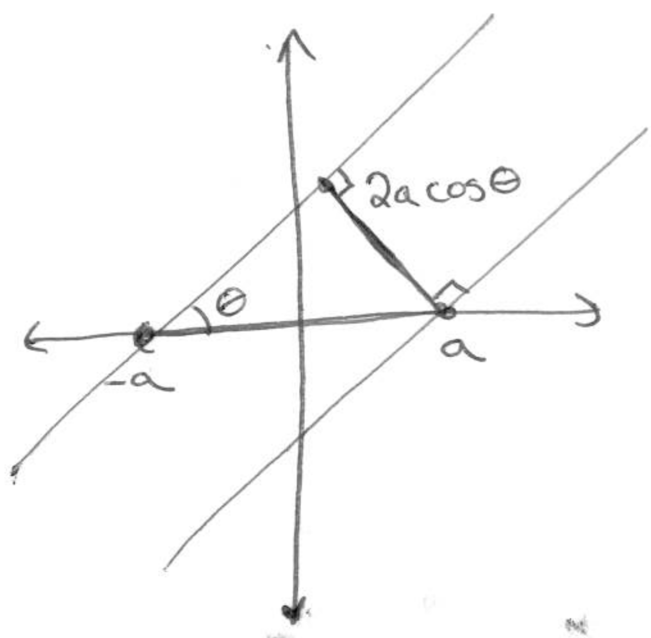
$$|d\Psi_{\theta_0,\vec{v}}| = 1$$

regardless of $\vec{v}, \theta_0$ or the mess in the upper right hand corner. We conclude

$$\int_L I_{\varphi(\alpha)}(p,\theta)\,dp\,d\theta = \int_L I_\alpha(p,\theta)\,dp\,d\theta$$

for any rigid motion φ of the plane.

Example. Let $\alpha = (-a,0)\,(a,0)$.



We see that

$$\int_L I_\alpha(p,\theta)\,dp\,d\theta = \int_{\theta} \left| \int_{-a\cos\theta}^{a\cos\theta} 1 \, dp \right| d\theta$$

$$= \int_{\theta} 2a\,|\cos\theta|\,d\theta = 4a \int_{-\pi/2}^{\pi/2} \cos\theta \, d\theta$$

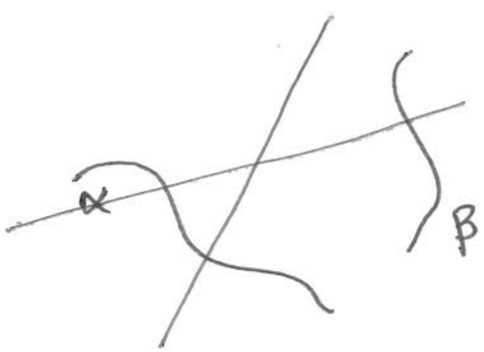
$$= 8a = 4 \cdot (\text{length of line segment}).$$

Since our integral is invariant under rigid motions of the line segment, this is true for any line segment in the plane.

We can now prove theorem using only one more idea

$$\int_L I_{\alpha+\beta}(p,\theta)\,dp\,d\theta = \int_L I_\alpha(p,\theta)\,dp\,d\theta + \int_L I_\beta(p,\theta)\,dp\,d\theta$$

for any pair of curves α, β, since



intersections with $\alpha+\beta$ = # intersections with α + # intersections with β.

So if we take a polygon, $\mathbb{P}$

$$\int_L I_{\mathbb{P}}(p,\theta)\,dp\,d\theta = 4\,\text{Length}(P)$$

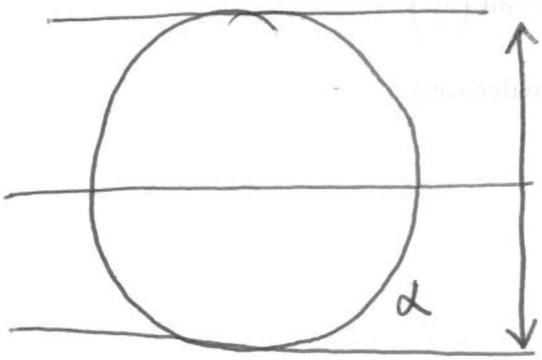
since $\mathbb{P}$ is a union of line segments.

But we can take $P_i \to \alpha$ so that $\text{Length}(P_i) \to \text{Length}(\alpha)$. Since our integrals

$$\int_L I_{P_i}(p,\theta)\,dp\,d\theta \to \int_L I_\alpha(p,\theta)\,dp\,d\theta$$

we have proved the theorem.

Example. The circle



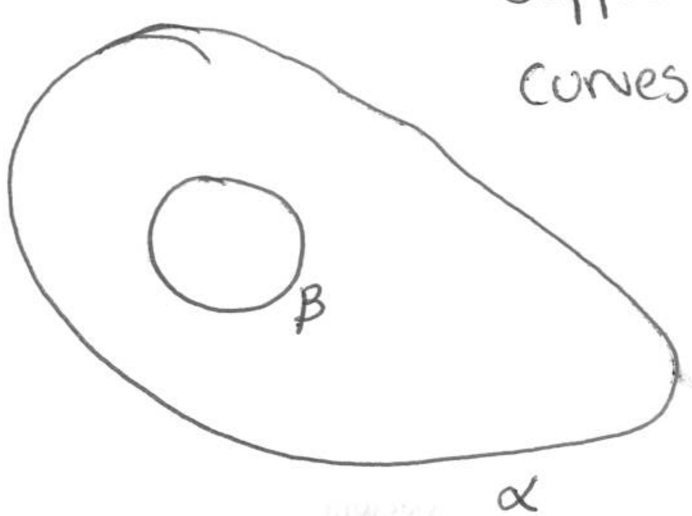
2R, 2 intersections per line

so

$$\int_L I_\alpha(p,\theta)\,dp\,d\theta = \int_\Theta 4R\,d\theta = 8\pi R = 4\underbrace{(2\pi R)}_{\text{Length}(\alpha)}.$$

Here is a neat application of this formula.

Suppose α, β are convex curves and $\beta \subset \alpha$.



Proposition. The probability that a random line through α intersects β is $\text{Length}(\beta)/\text{Length}(\alpha)$.

Corollary. $\text{Length}(\beta) < \text{Length}(\alpha)$.

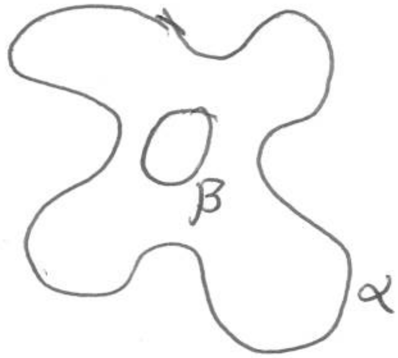
We Know that a line intersecting a convex curve intersects it exactly twice, ~~So~~ and that every line through β hits α, so

$$\frac{\text{Volume (lines through } \beta)}{\text{Volume (lines through } \alpha)} = P(\ell \text{ intersects } \beta \text{ given } \ell \text{ int. } \alpha)$$

and this is just $\text{Length}(\beta)/\text{Length}(\alpha)$ by our formula.

Since ~~the~~ ~~number~~ every line through β intersects α, this fraction is ≤ 1.

Corollary 2. If α is <u>any</u> plane curve enclosing a convex curve β, then $\text{Length}(\alpha) > \text{Length}(\beta)$.



Proof. Any line through β intersects β exactly twice and α at least twice. $\square$

We can call the Proposition the "egg yolk lemma" because it gives the probability of puncturing the yolk with a ~~fen~~ needle inserted randomly into an egg.