Introduction to Integral Geometry

We now change viewpoint for one last time before switching our interest from curves to surfaces.

We are interested in measuring geometric properties of curves; previously we did this by carefully analyzing local information. Now we'll do this by averaging <u>global</u> information!

Theorem 1. (Integralgeometric measure)

Given a direction $\vec{p} \in S^1$, let $\text{len}_p\, \alpha$ be the length of the projection of α to $\vec{p}$. ~~to the direction~~

We have

$$\text{Len}\ \alpha = \frac{1}{4} \int_{\vec{p}(\theta) \in S^1} \text{Len}_{p(\theta)}\ \alpha \ d\theta .$$

That is, the length of a curve is the average length of its projections onto all lines through the origin, multiplied by some constant factor.

Proof. We can write the projection of α to $\vec{p}$ as $\vec{p} \cdot \vec{\alpha}(s)$. We observe

$$\mathrm{Len}_{\vec{p}}\, \vec{\alpha} = \int_{s=0}^{\ell} \left| \vec{p} \cdot \vec{\alpha}(s)' \right| ds$$

$$= \int_{s=0}^{\ell} \left| \vec{p} \cdot \vec{\alpha}'(s) \right| ds$$

$$= \int_{s=0}^{\ell} |\vec{p}|\, |\vec{\alpha}'(s)|\, |\cos\theta|\, ds,$$

where θ is the angle between $\vec{\alpha}'(s)$ and $\vec{p}$. Now as $\vec{p}$ varies around S^1, θ varies from 0 to 2π

$$\int_{\theta=0}^{2\pi} \int_{s=0}^{\ell} |\cos\theta|\, ds\, d\theta = \int_0^{\ell} \int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta + \int_{\pi/2}^{3\pi/2} -\cos\theta\, d\theta\, ds$$

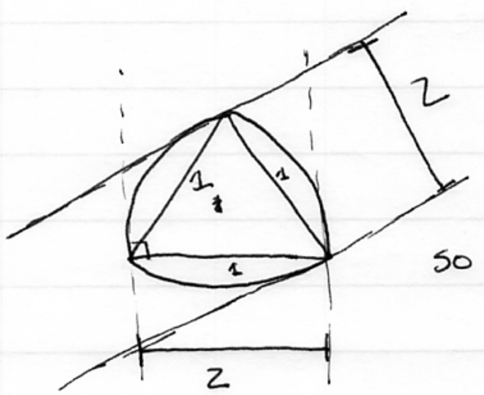
$$= \int_0^{\ell} \sin\theta \Big|_{-\pi/2}^{\pi/2} - \sin\theta \Big|_{\pi/2}^{3\pi/2} ds$$

$$= \int_0^{\ell} 2 - (-2)\, ds = 4\ell$$

Thus $\frac{1}{4} \int_{\theta=0}^{2\pi} \mathrm{Len}_p\, \alpha\, d\theta = \ell$, as desired. $\therefore$

Example curves.

so $\frac{1}{4}\int_0^{2\pi} 4 \, d\theta = 2\pi = \text{Length } \alpha.$ ✓



so $\frac{1}{4}\int_0^{2\pi} 2 \, d\Theta = \pi.$ Len $\alpha = 3 \cdot \frac{2\pi}{6} = \pi.$ ✓

$$\int_0^{2\pi}\int_0^{2\pi} \sqrt{a\cos t\cos\theta + b\sin t\sin\theta} \, dt \, d\theta$$

$$\overset{?}{=}$$

$\alpha(t) = (a\cos t, b\sin t)$

$\vec{p}(\theta) = (\cos\theta, \sin\theta)$