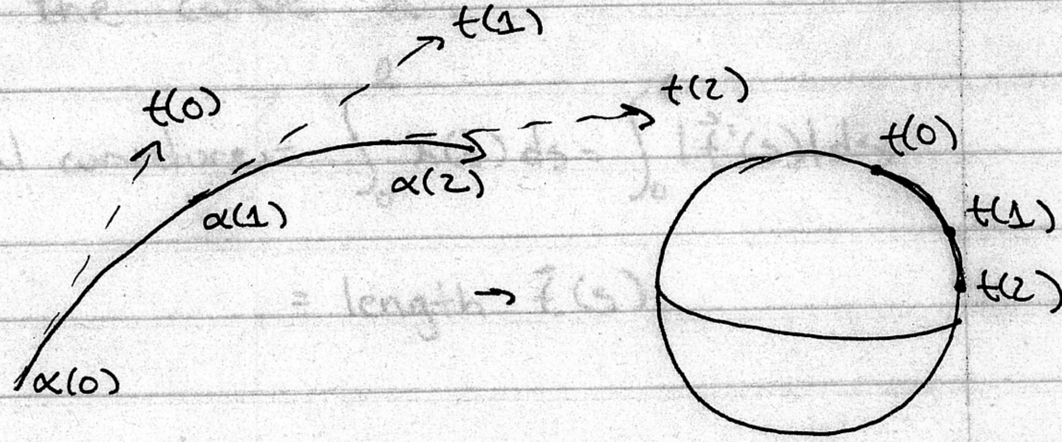
The Four-Vertex Theorem and beyond!

We begin by introducing a key idea in the study of curves.

Definition. The <u>tangent</u> <u>indicatrix</u> of a unit-speed parametrized differentiable curve $\alpha(s)$ is the curve $t(s) = \alpha'(s)$.

Note that the tangent indicatrix, or <u>tantrix</u> is a curve on the unit sphere.



Even though $\alpha(s)$ is unit speed, $t(s)$ probably is not. In fact, Proposition. The velocity $|\vec{t}\,'(s)|$ of the tantrix is the curvature $\kappa(s)$ of α at s.

Proof. This is pretty much the <u>definition</u> of curvature.

~~For a plane curve,~~

We can write down the <u>length</u> of the tantrix as the <u>total</u> <u>curvature</u> of the curve α:

$$\text{total curvature} = \int_0^{\ell} \kappa(s)\, ds = \int_0^{\ell} |\vec{t}\,'(s)|\, ds$$

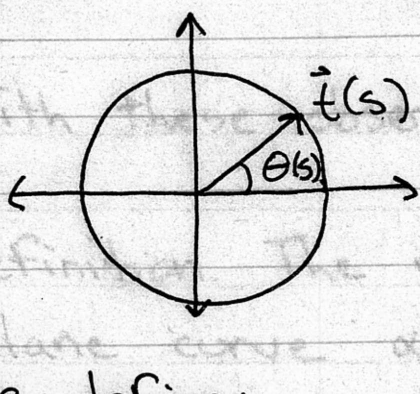
$$= \text{length } \vec{t}(s).$$

For a plane curve, we can be more specific; $\vec{t}(s)$ lies on the unit circle. ~~and we know~~

~~Proposition. The total curvature of a closed plane curve is a multiple of 2π.~~

~~Proof. The tantrix of α must also be a closed curve on the unit circle; all such curves have length a multiple o~~

We can describe the position of $\vec{t}(s)$ on the unit circle by some angle $\theta(s)$



~~We define:~~

~~Definition. The rotation index of a plane curve is given by~~

$$\cancel{I = \frac{1}{2\pi}\int}$$

While $\theta(s)$ is defined only up to a multiple of 2π by the above, we can certainly define $\theta(0)$ to be between 0 and 2π, and then extend this definition to all $\theta(s)$ by observing

a) $\theta'(s)$ is well-defined

b) $\theta(s) = \int_0^s \theta'(x)\,dx + \theta(0)$ defines $\theta(s)$ uniquely.

With these observations, we can write

Definition. The <u>rotation index</u> of a plane curve $\alpha(s)$ is given by

$$I = \frac{1}{2\pi}\left(\theta(\ell) - \theta(0)\right)$$

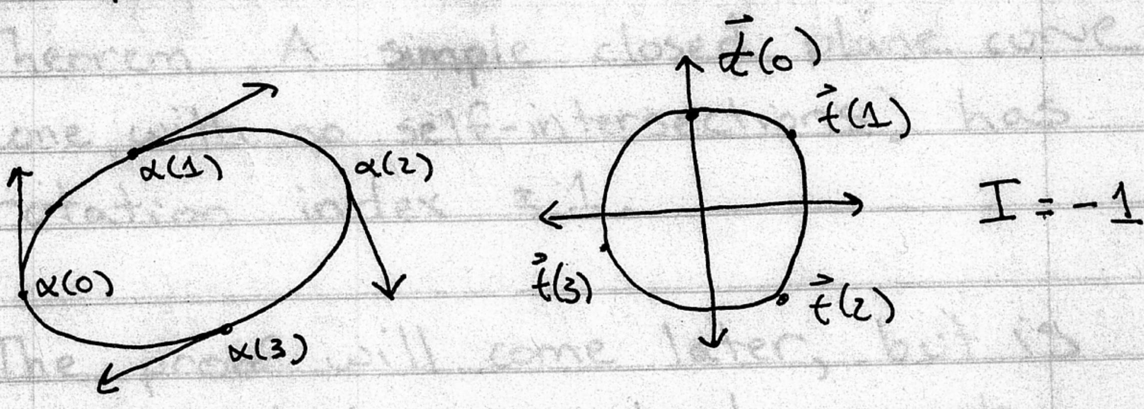
where ℓ is the length of α.

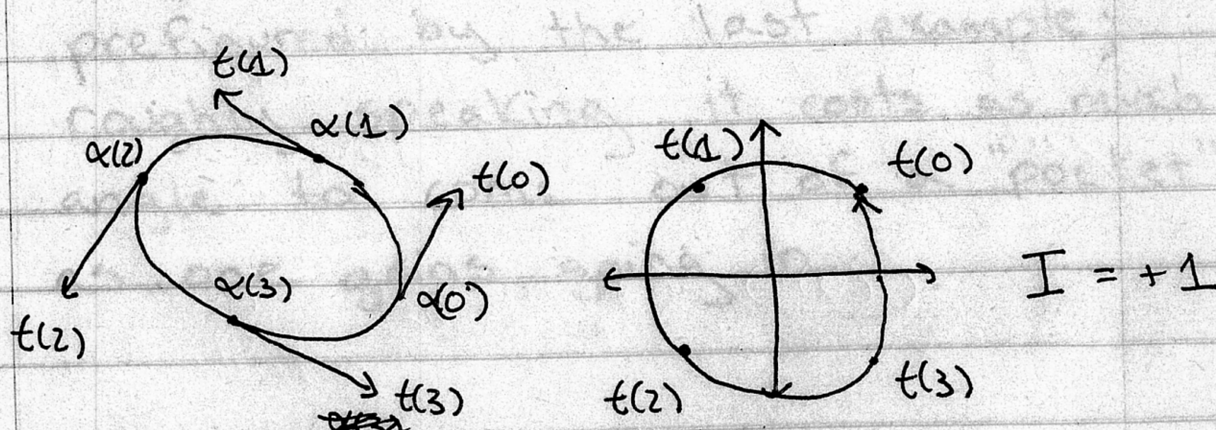
Notice that

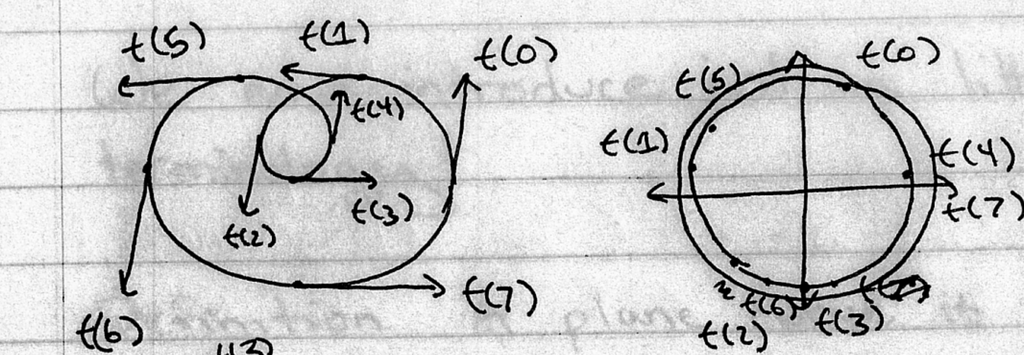
Lemma. If $\alpha(s)$ is a <u>closed</u> plane curve its rotation index I is an integer.

Proof. $\vec{t}(s)$ must also be closed; thus $\theta(0)$ and $\theta(\ell)$ differ by a multiple of 2π.

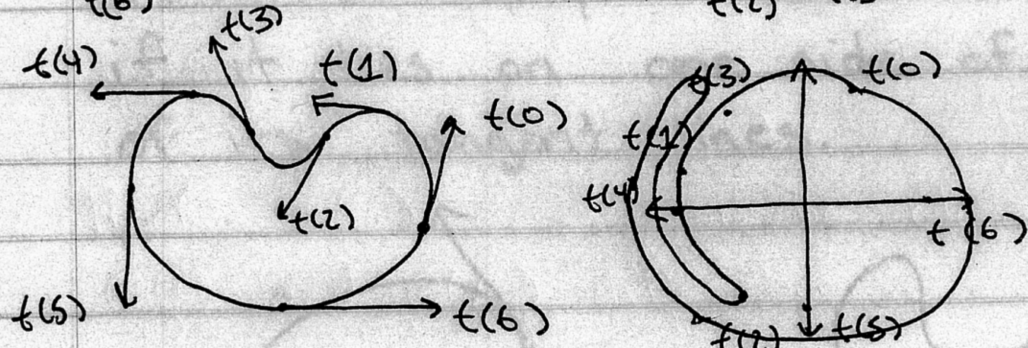
Examples.







$I = +2$



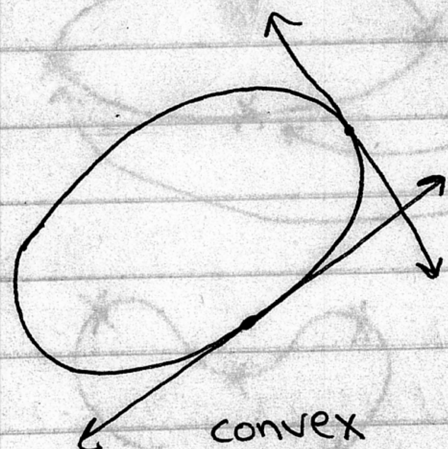
$I = +1$

Theorem. A simple closed plane curve (one with no self-intersections) has rotation index ± 1.

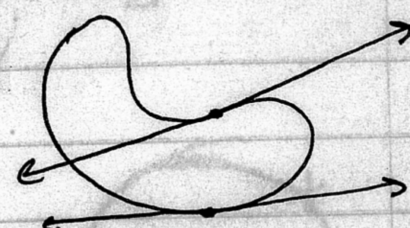
The proof will come later, but is prefigured by the last example; roughly speaking it costs as much angle to come out of a "pocket" as one gains going in.

We now introduce just a little more terminology!

Definition. A plane curve is convex if it lies on one side of each of its tangent lines.

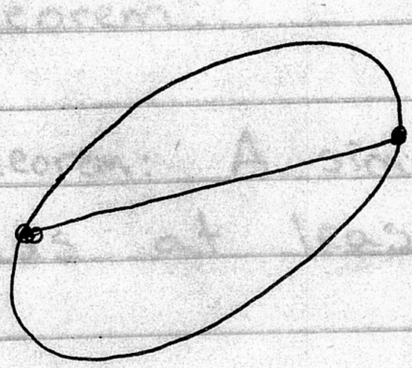


convex

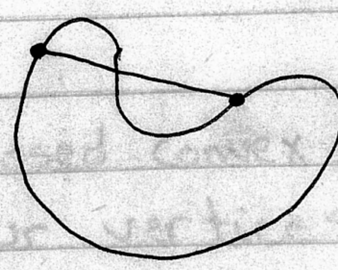


not convex

Equivalently (though we won't prove it) a curve is convex if any two points on the curve can be connected by a line inside the curve



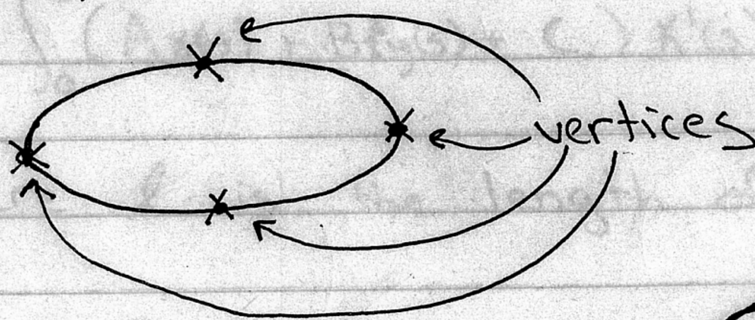
convex

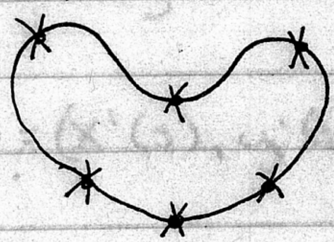


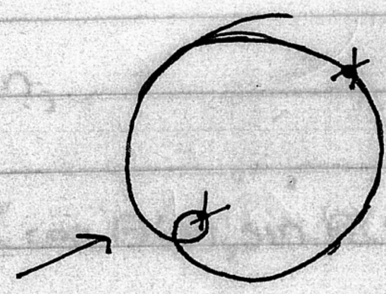
not convex

Definition. A <u>vertex</u> of a plane curve $\alpha(s)$ is a point where $\kappa'(s) = 0$ (a critical point of curvature).

Examples







Note that this answers our challenge problem from the first class!

We now start on an amazing theorem:

Theorem: A simple closed convex curve has at least four vertices.

Before starting, we need a lemma.

Lemma. If $\alpha(s) = (x(s), y(s))$, $\alpha'(s) = (\cos\theta(s), \sin\theta(s))$, and A, B, C are real numbers, then

$$\int_0^{\ell} (A\,x(s) + B\,y(s) + C)\,\theta''(s)\,ds = 0.$$

(as usual, ℓ is the length of the curve α).

Proof. Observe that

$$x'(s) = \cos\theta(s) \qquad y'(s) = \sin\theta(s)$$

So

$$\begin{aligned} x''(s) &= -\sin\theta(s)\,\theta'(s) \\ &= -y'(s)\,\theta'(s) \end{aligned} \qquad \begin{aligned} y''(s) &= \cos\theta(s)\,\theta'(s) \\ &= x'(s)\,\theta'(s). \end{aligned}$$

Now we can play a game! Since $x''(s)$ and $y''(s)$ are periodic functions of s, and $x'(s)$, $y'(s)$ are as well,

$$\int_\alpha x''(s)\,ds = \int_\alpha y''(s)\,ds = 0.$$

But we know

$$\int_0^\ell x''(s)\,ds = \int_0^\ell -y'(s)\,\theta'(s)\,ds$$

Let's do the next step in slow motion:

$$\frac{d}{ds}\, y(s)\,\theta'(s) = y'(s)\,\theta'(s) + y(s)\,\theta''(s)$$

Integrating both sides from 0 to ℓ, we see

$$y(\ell)\,\theta'(s) - y(0)\,\theta'(0) = \int_0^\ell y'(s)\,\theta'(s)\,ds + \int_0^\ell y(s)\,\theta''(s)\,ds$$

Now both $y(s)$ and $\theta'(s)$ are periodic since α is a closed curve! Thus

$$-\int_0^\ell y'(s)\,\theta'(s)\,ds = \int_0^\ell y(s)\,\theta''(s)\,ds$$

Integration by parts!

Thus

$$\int_0^\ell y(s)\,\theta''(s)\,ds = 0.$$

We can play exactly the same game with the other term

$$0 = \int_0^\ell y''(s)\,ds = \int_0^\ell x'(s)\,\theta'(s)\,ds$$

$$= -\int_0^\ell x(s)\,\theta''(s)\,ds.$$

Now we (last) observe that

$$\int_0^\ell \theta''(s)\,ds = \theta'(\ell) - \theta'(0)$$

$$= 0.$$

But these three integrals are the components of our original integral!

$$\int_0^\ell (A x(s) + B y(s) + c)\,\theta''(s)\,ds$$

$$= A\int_0^\ell x(s)\theta''(s)\,ds +$$

$$B\int_0^\ell y(s)\,\theta''(s)\,ds +$$

$$c\int_0^\ell \theta''(s)\,ds. = 0.$$

This proves the lemma. $\therefore$

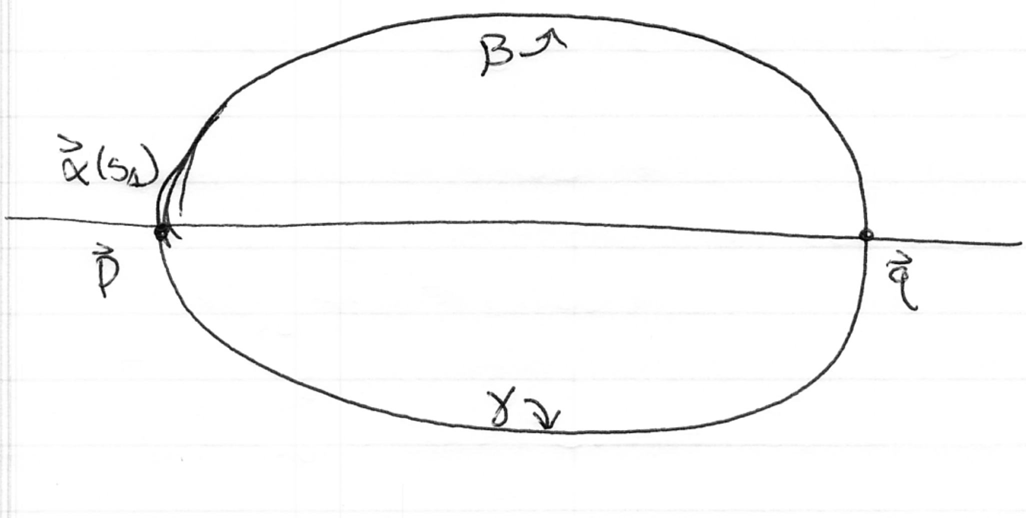
We now prove the theorem!

Proof. Let $\vec{\alpha}(s)$ be parametrized by arclength on $[0, \ell]$. Since $\kappa(s)$ is continuous, it reaches a maximum and minimum value on $[0, \ell]$.

These points are vertices of α, at

$$\vec{\alpha}(s_1) = \vec{p} \quad \text{and} \quad \vec{\alpha}(s_2) = \vec{q}.$$

Connect these with a line L, and let β and γ be the arcs of α ~~on either side of L;~~ determined by these points.



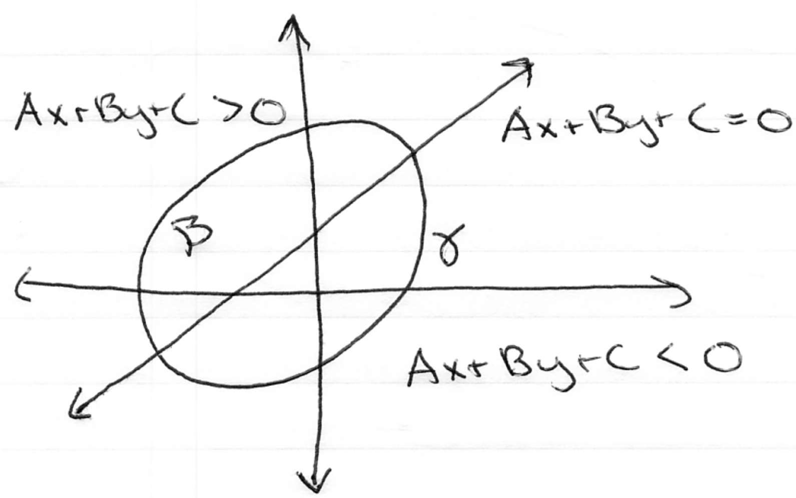
Now we claim β and γ lie on opposite sides of L; this follows from the convexity of α.

~~Now~~ We now want to play another game: ~~For~~ we observe that $\Theta''(s) = K'(s)$. Further, ~~at~~ at a vertex of α, $K'(s) = \Theta''(s) = 0$.

Now if there are no other vertices on α, $K'(s)$ must have one sign on β and the other sign on γ.

We really use convexity!

Further, if $Ax + By + C = 0$ is the equation of L, then



So we have (choosing the sign of A, B, C if we have to) the estimate

$$(Ax(s)+By(s)+C)\,K'(s) > 0$$

on both β and γ. But then

$$\int_0^\ell (A\,x(s)+B\,y(s)+C)\,K'(s) > 0.$$

But $K'(s) = \theta''(s)$, and we just showed

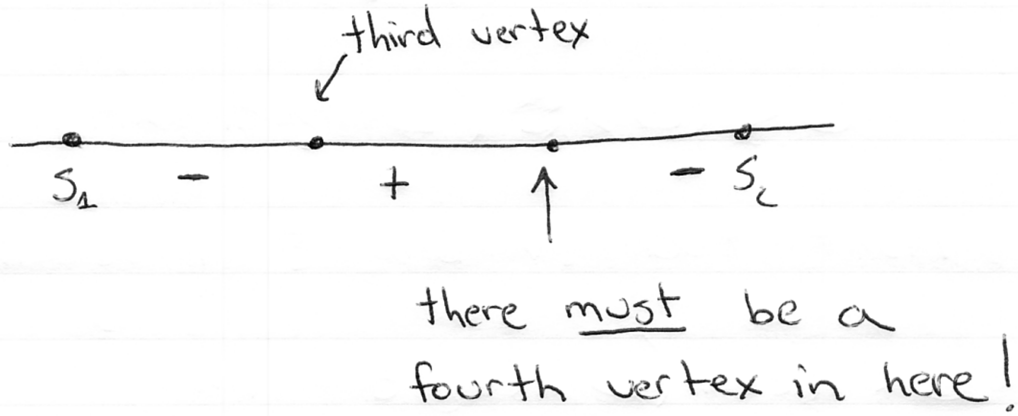
$$\int_0^\ell (A\,x(s)+B\,y(s)+C)\,\theta''(s) = 0. \quad \times$$

Thus the function $K'(s)$ <u>must</u> change sign on β or γ, creating a <u>third</u> vertex! Suppose this happens on γ; we have just shown that the function $K(s)$ has a local max or local min on $[s_1, s_2]$.

It's a good thing we picked the global max and min!

But s_1 is the global max and s_2 the global min of $K(s)$, so $K'(s)$ must have the <u>same</u> sign near s_1 and s_2 - negative.

Drawing the plot of maxes and mins from 2200, we see



This completes the proof!

The theorem is still true for simple (but non-convex) curves; but the proof is harder!

Is there a converse? Yes!

The rotation index is too much fun to pass up.