Link, Twist, and Writhe

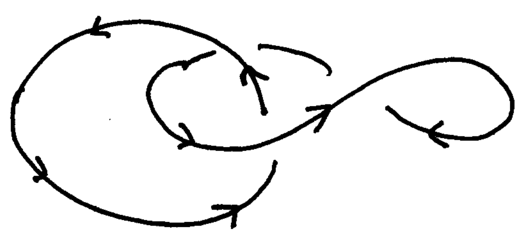
We introduced an interesting guy last time: the <u>twist</u> of a frame on a curve in $\mathbb{R}^3$. Recall that

$$Tw(V,\alpha) = \frac{1}{2\pi}\int V' \cdot (\alpha' \times V)\, ds$$

and it measures the rate at which the frame is spinning around the tangent vector.

Twist is part of a larger story in the global geometry of curves.

Definition. A link is a collection of closed, parametrized curves in $\mathbb{R}^3$.



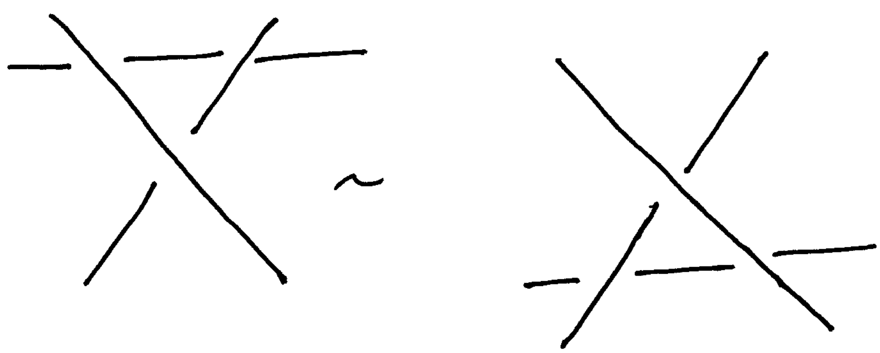
The individual curves are called the <u>components</u> of the link.

The collection of links that can be deformed to L without crossings is the *link type* of L.

It is the case that any two diagrams of links in the same link type can be transformed into one another by a sequence of "Reidemeister moves".

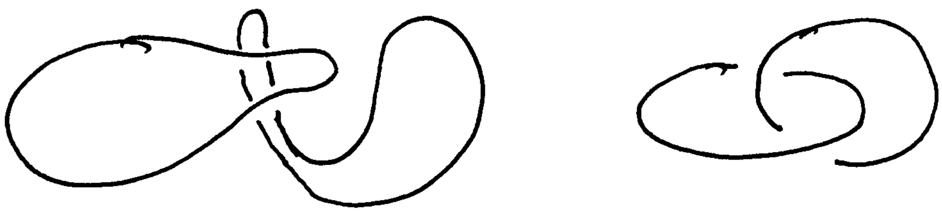
I

II



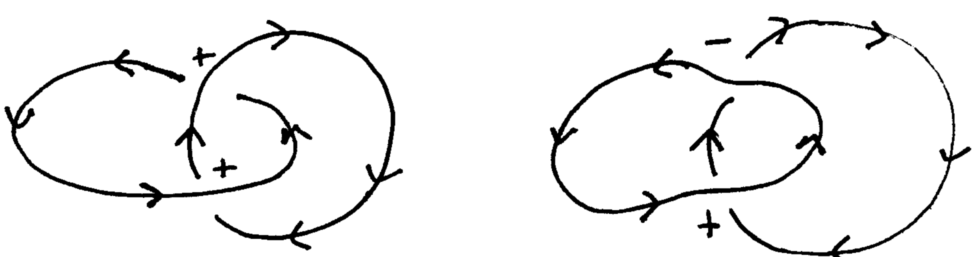
III

Given two components of a link, how can we tell whether they can be separated?



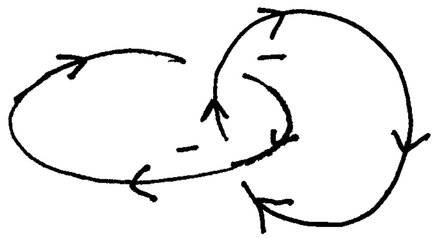
Definition. A crossing is positive or negative according to the convention below

Definition. The <u>linking</u> <u>number</u> of L_1, L_2 is the sum of the signs of the crossings of L_1 with L_2, divided by 2.



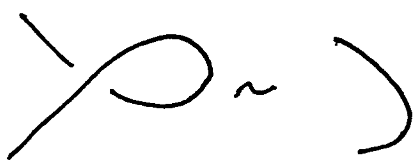
$LK(L_1, L_2) = \frac{1}{2}(1+1) = 1$ $\quad$ $LK(L_1, L_2) = \frac{1}{2}(1-1) = 0.$

Notice that the sign of $LK(L_1, L_2)$ depends on the orientation of the curves.



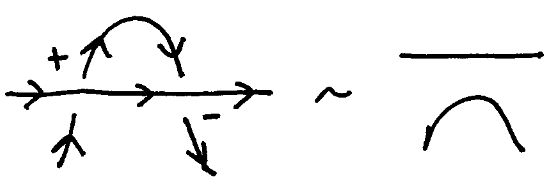
$LK(L_1, L_2) = \frac{1}{2}(-1-1) = -1$

Theorem. The linking number is invariant under Reidemeister moves.

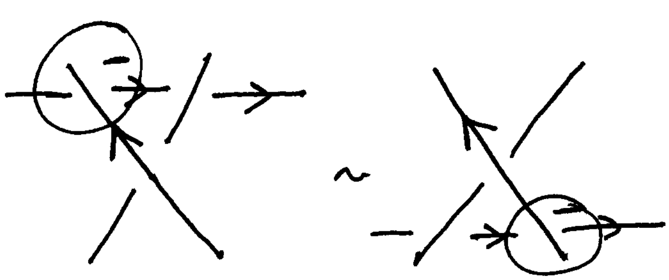


I

nothing to check, since this crossing is not between different components



crossings at left have opposite signs and do not contribute



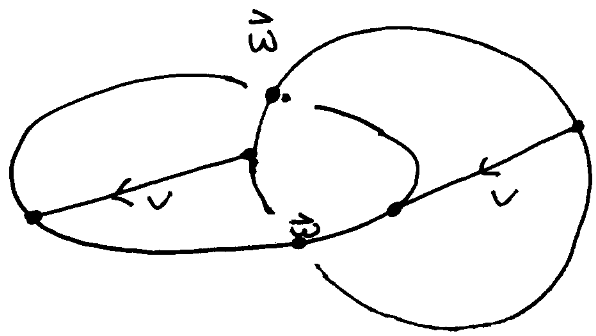
circled crossings have the same sign

So components which have $LK \neq 0$ can't be separated from each other.

This allows us to redefine LK as an integral.

Proposition. $LK(L_1, L_2)$ is the average signed crossing number of L_1, L_2.

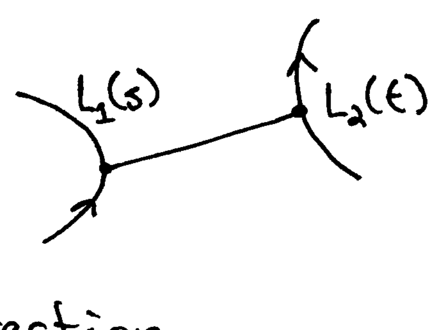
$$LK(L_1,L_2) = \frac{1}{4\pi}\int_{v\in S^2} \begin{matrix}\text{Crossing \# viewed in}\\ \text{direction } \vec{v}\end{matrix}\ d\text{Area}_{S^2}$$



We see different crossings from different directions.

Proof. We are averaging a constant (by the theorem before).

Let's rewrite this integral in a clever way. Every pair of points on the two curves is a crossing from some direction:

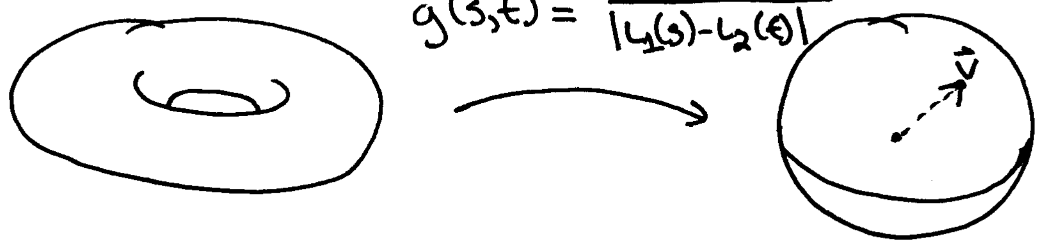


a crossing from the $\dfrac{L_1(s)-L_2(t)}{|L_1(s)-L_2(t)|}$ direction.

So instead of counting crossings from each projection on S^2, let's count projection directions from each point on the product

$$L_1 \times L_2 = S^1 \times S^1 = T^2$$

$$g(s,t) = \frac{L_1(s) - L_2(t)}{|L_1(s) - L_2(t)|}$$

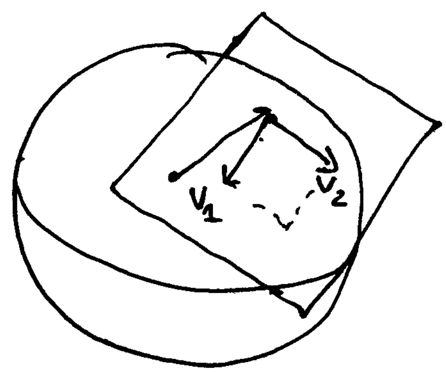


We can integrate over T^2 instead of S^2 by "pulling back the area form" on the sphere.

discuss forms and area ←

$$d\text{Area}(V_1, V_2) = |\vec{V}_1 \times \vec{V}_2|$$

↑ area of parallelogram spanned by V_1, V_2

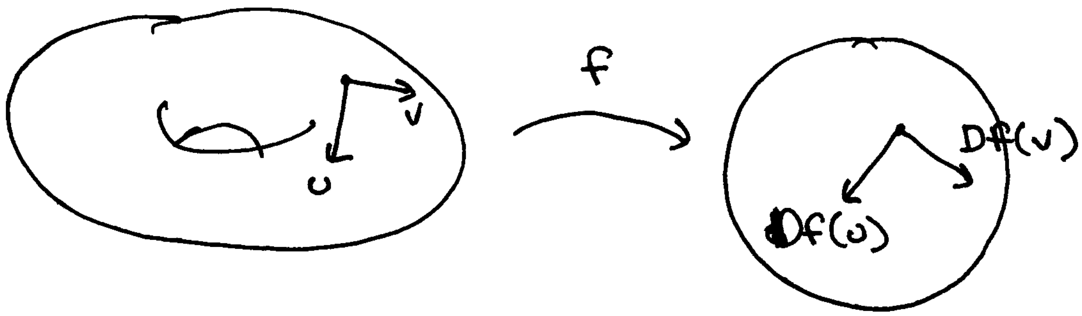


At a point $\vec{p}$ on the sphere, V_1 and V_2 are tangent to the sphere, so

$$|\vec{V}_1 \times \vec{V}_2| = \vec{V}_1 \times \vec{V}_2 \cdot \vec{p}.$$

To "pull back" a form, we use the differential or Jacobian of the map between surfaces.

Idea:



$f^*\omega$ eats pairs of vectors in bilinear way, returns #s

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$$f^*\omega(u,v) = \omega(Df(u), Df(v))$$

Let's try an example

$$Dg = \begin{bmatrix} \frac{\partial g_1}{\partial s} & \frac{\partial g_1}{\partial t} \\ \vdots & \vdots \\ \frac{\partial g_3}{\partial s} & \frac{\partial g_3}{\partial t} \end{bmatrix}$$

$$\frac{\partial g}{\partial s} = \frac{\partial}{\partial s} \frac{L_1(s) - L_2(t)}{|L_1(s) - L_2(t)|}$$

$$= \frac{L_1'(s)}{|L_1(s) - L_2(t)|} + \left(\frac{\partial}{\partial s}\frac{1}{|\quad|}\right) L_1 - L_2$$

So we write $\int_{T^2} g^*(d\text{Area})$

$$\int_{S^2} (\#\text{ of crossings})\, dA = \int_{T^2} \left(\frac{L_1'}{|L_1-L_2|} + \left(\frac{\partial}{\partial s}\frac{1}{|L_1-L_2|}\right) L_1-L_2 \right)$$

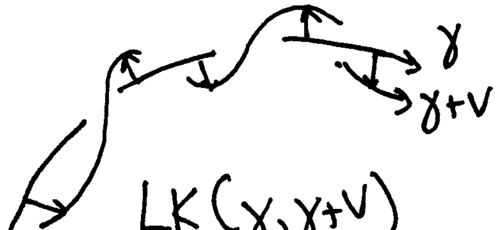
$$\times \left(\frac{L_2'}{|L_1-L_2|} + \left(\frac{\partial}{\partial t}\frac{1}{|L_1-L_2|}\right) L_1-L_2 \right) \cdot \frac{L_1-L_2}{|L_1-L_2|}\, ds\, dt$$

Since the L_1-L_2 components are all colinear, this is equal to

$$= \int_{T^2} \frac{L_1' \times L_2' \cdot (L_1-L_2)}{|L_1-L_2|^3}\, ds\, dt$$

This is the Gauss integral for linking #! It is rather surprising that you can compute something like this by doing an integral.

Idea: There ought to be a relation between



and

$LK(\gamma, \gamma+V)$

average crossing # of γ by itself

This guy is called "Writhe":

$$Wr(\gamma) = \int_{T^2} \frac{\gamma'(s) \times \gamma'(t) \cdot (\gamma(s) - \gamma(t))}{|\gamma(s)-\gamma(t)|^3} \, ds\, dt$$

It is not an integer, nor is it invariant under Reidemeister moves...

$\sim$

-1 (from this direction) $\qquad$ 0 (from another direction).

$+1$

$Wr \approx +1$ $\leftarrow$

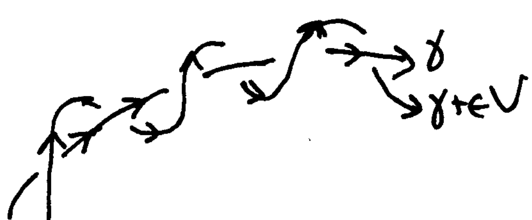
Natural conjecture:

-1

$Wr \approx -1$

$$\lim_{\epsilon \to 0} LK(\gamma, \gamma + \epsilon V) = Wr(\gamma)$$

But this is <u>false</u>!



$LK = -3$

$Wr = 0$

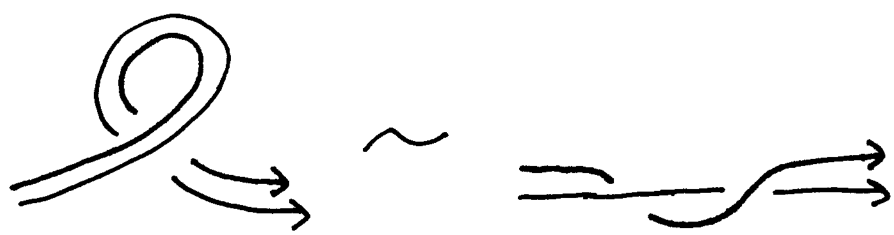
In fact, we have

Theorem [Calugareanu]

$$\cancel{Lk = Wr + Tw}$$

$$LK(\gamma, \gamma + \epsilon V) = Tw(\gamma, V) + Wr(\gamma).$$

Beautiful pictures:



Wr, but no Tw   Tw, but no Wr

(Demonstration with elastic rod)