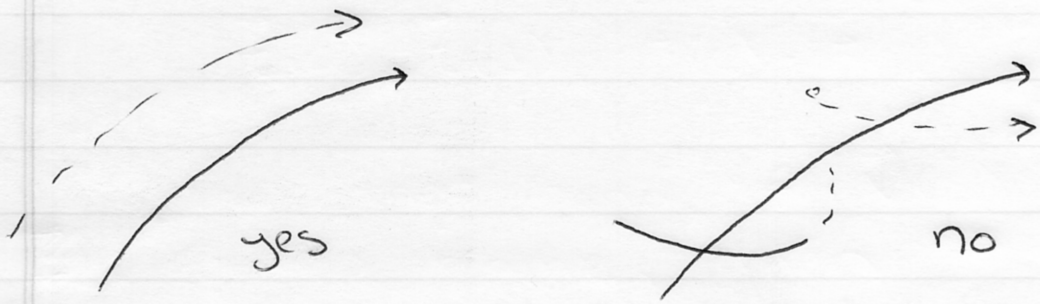
Differential Geometry L3/4

What about curves where Frenét vanishes?

The Bishop Frame.

Definition. A normal vector field $\vec{V}(s)$ on $\vec{\alpha}(s)$ is called <u>relatively parallel</u> if it's derivative is parallel to the tangent vector of $\vec{\alpha}$.



Observe that:

Lemma. Any relatively parallel field $\vec{V}(s)$ has constant length.

Proof. Our favorite proof in reverse;
$\vec{V}'(s) \cdot \vec{V}(s) = 0$ since $\vec{V}(s) \cdot \vec{\alpha}'(s) \equiv 0$.

Now observe:

Proposition. Any choice of $\vec{v}(0)$ normal to $\vec{\alpha}'(0)$ generates a <u>unique</u> relatively paralled frame $\vec{v}(s)$.

Proof.

Uniqueness: Any two $\vec{V}(s), \vec{W}(s)$ which are both relatively parallel have $\vec{V}(s)-\vec{\omega}(s)$ relatively parallel, and hence constant length. But $\vec{V}(0)-\vec{\omega}(0)=\vec{0}$ by assumption; so $\vec{V}(s)-\vec{\omega}(s)=\vec{0}$ everywhere.

Existence: This is a little harder, but not much.

Take any smooth normal framing $\vec{\omega}_1(s), \vec{\omega}_2(s)$ on $\alpha(s)$. We observe that if

$$\vec{V}(s) = \cos\theta(s)\,\vec{\omega}_1(s) + \sin\theta(s)\,\vec{\omega}_2(s)$$

then θ obeys the differential equation:

well,

$$V'(s) = -\sin\theta(s)\ \theta'(s)\ \vec{\omega}_1(s)$$
$$\cos\theta(s)\ \vec{\omega}_1{}'(s)$$
$$\cos\theta(s)\ \theta'(s)\ \vec{\omega}_2(s)$$
$$\sin\theta(s)\ \vec{\omega}_2{}'(s).$$

But just like the Frenet frame:

$$
\begin{aligned}
T'(s) &= \qquad p_{01}(s)\,\vec{\omega}_1(s) + p_{02}(s)\,\vec{\omega}_2(s)\\
\omega_1'(s) &= -p_{01}(s)\,T(s) \qquad\qquad p_{12}(s)\,\vec{\omega}_2(s)\\
\omega_2'(s) &= -p_{02}(s)\,\vec{T}(s) - p_{12}(s)\,\vec{\omega}_1(s)
\end{aligned}
$$

for some guys p_{01}, p_{02}, p_{12}. Thus

$$V'(s) = \cancel{(-\theta'(s)\sin(s)-}$$

$$
\begin{aligned}
&\left(-\sin\theta(s)\ \theta'(s) - \sin\theta(s)\ p_{12}(s)\right)\vec{\omega}_1\\
+&\left(\cos\theta(s)\ \theta'(s) + \cos\theta(s)\ p_{12}(s)\right)\vec{\omega}_2\\
+&\left(\qquad\qquad\qquad\qquad\right)\vec{T}.
\end{aligned}
$$

$$
\begin{aligned}
= &\left(p_{12}(s) + \theta'(s)\right)\left(-\sin\theta(s)\ \vec{\omega}_1 + \cos\theta\ \vec{\omega}_2\right).\\
&+\left(\qquad\qquad\qquad\right)\vec{T}
\end{aligned}
$$

This gives us a very simple

ODE for Θ:

$$\Theta'(s) = -p_{12}(s).$$

Of course, geometrically

$$p_{12}(s) = \vec{\omega}_1'(s) \cdot \vec{\omega}_2(s).$$

L4

Done

This has another interpretation: we define

Definition. The twist of a normal vector field $\vec{V}(s)$ is defined by

$$Tw(\vec{V}(s), \vec{\alpha}(s)) = \frac{1}{2\pi} \int \vec{V}'(s) \cdot (\alpha'(s) \times V(s))\, ds.$$

The <u>twist rate</u> of $\vec{V}$ is just the integrand above.

Examples:

Frenet frame has twist rate $= \tau(s)$
General frame has twist rate $= p_{12}(s)$
Relatively parallel frame has twist rate 0.

We now write down Frenet equations for a relatively parallel frame $\vec{T}, \vec{V}_1, \vec{V}_2$

$$\begin{aligned} T'(s) &= \qquad K_1 V_1 + K_2 V_2 \\ V_1'(s) &= -K_1 T \\ V_2'(s) &= -K_2 T \end{aligned}$$

The entries K_1 and K_2 will be sort of like curvature and torsion. In fact

$$|T'| = \kappa(s) = \sqrt{K_1^2 + K_2^2}$$

so in the κ_1, κ_2 plane, curvature is a sort of radius. Torsion will be a little harder to grasp: observe

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(\frac{K_1}{\kappa}\right) V_1 + \left(\frac{K_2}{\kappa}\right) V_2$$

Now if we write $\frac{\kappa_1}{\kappa} = \cos\theta$, $\frac{\kappa_2}{\kappa} = \sin\theta$,

then we obsefve that θ' is the <u>twist</u>

rate of $\vec{N}(s)$ in this frame. But that twist rate is already torsion, so

$$\Theta = \int \tau(s)\,ds$$

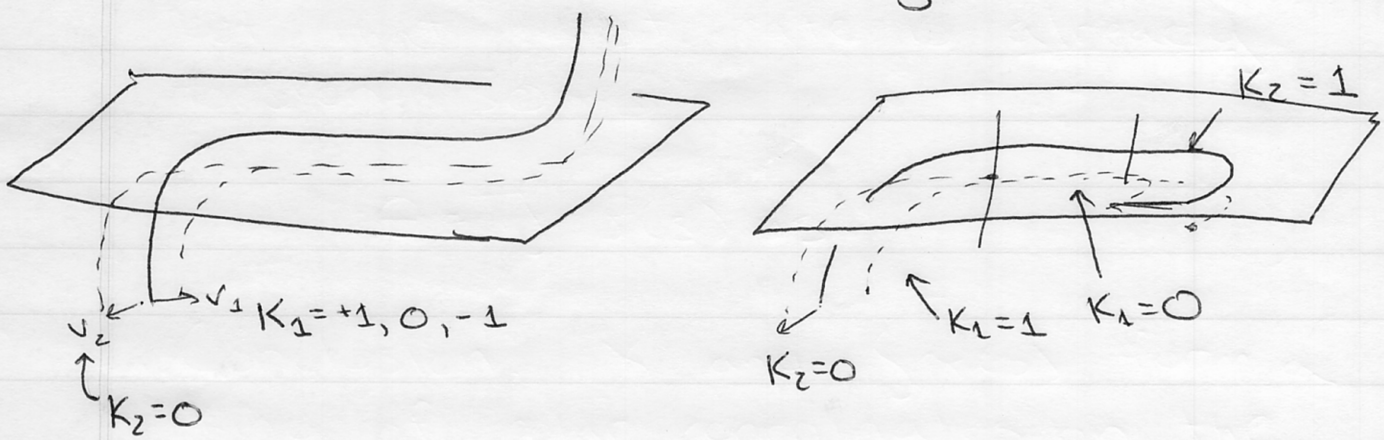
Conclusion: κ and $\int \tau(s)$ form a kind of polar coordinates on the κ_1, κ_2 plane.

Definition: We call the ~~κ_1~~ curve κ_1, κ_2 the normal development of α.

~~Theorem. Two curves α and $\bar{\alpha}$ with relatively parallel frames and the same κ_1, κ_2~~

Theorem. If α and $\bar{\alpha}$ are regular C^2 curves with relatively parallel frames with coefficients κ_1, κ_2 and $\bar{\kappa}_1, \bar{\kappa}_2$ then α may be superimposed on $\bar{\alpha}$ by a rigid ~~rotat~~ motion of $\mathbb{R}^3$ $\Leftrightarrow$ the curves (κ_1, κ_2) and $(\bar{\kappa}_1, \bar{\kappa}_2)$ may be superimposed on one another by a rigid rotation of $\mathbb{R}^2$.

Now we've beat the system:



Thus only curves where the <u>tangent</u> vanishes can fool the Bishop frame. (Of course, those are pretty hard to frame in the first place!)

Neat application.

A C^2 regular curve is spherical $\iff$ its normal development is a line not through the origin.

Proof. Suppose α lies on a sphere of center P and radius r. Then

$$(\alpha(s) - P) \cdot (\alpha(s) - P) = r^2$$

Differentiating,

$$\alpha'(s) \cdot (\alpha(s) - P) = 0,$$

so for an RPAF V_1, V_2 on $\alpha(s)$,

$$\alpha(s) - P = f(s)\, V_1(s) + g(s)\, V_2(s)$$

But

$$f'(s) = \frac{d}{ds}(\alpha(s) - P) \cdot V_1(s)$$

$$= \alpha'(s) \cdot V_1(s) + (\alpha(s) - P) \cdot V_1'(s)$$

$$= 0$$

and g is constant, too.

We now take

$$\frac{d}{ds}\, \alpha'(s) \cdot (\alpha(s) - P)$$

$$= (K_1(s)\, V_1(s) + K_2(s)\, V_2(s)) \cdot (\alpha(s) - P)$$

$$+ \alpha'(s) \cdot \alpha'(s)$$

$$= f K_1(s) + g K_2(s) + 1 = 0.$$

Of course, this is exactly the equation of a line!

The converse is an exercise!