Curves of constant curvature and torsion

We have now defined curvature and torsion for arbitrary parametrizations of a curve. Of course, our description isn't all that useful, since we must reparametrize by arclength to compute the values of κ and τ.

We now present formulae for $\kappa(t)$ and $\tau(t)$.

$$\kappa(t) = \frac{|\gamma'(t) \times \gamma''(t)|}{|\gamma'(t)|^3}$$

$$\tau(t) = \frac{-(\gamma'(t) \times \gamma''(t)) \cdot \gamma'''(t)}{|\gamma'(t) \times \gamma''(t)|^2}$$

Proving these is a homework assignment, but here's a hint:

$$\frac{d}{dt}\, \gamma(s(t)) = \gamma'(s) \cdot s'(t)$$

where $s'(t) = |\gamma'(t)|$, as we saw a moment ago at the end of lecture 2. Thus

$$\gamma'(s) = \frac{\gamma'(t)}{|\gamma'(t)|}$$

Thus we have

$$\frac{d^2}{dt^2}\gamma''(s(t)) = \frac{d}{dt}\left(\gamma'(s)\cdot s'(t)\right)$$

$$= \gamma''(s)\cdot(s'(t))^2 + \gamma'(s)\cdot s''(t)$$

Here $s'(t) = |\gamma'(t)|$ so

$$s''(t) = \frac{1}{2}\langle\gamma'(t),\gamma'(t)\rangle^{-1/2}\cdot 2\langle\gamma'(t),\gamma''(t)\rangle$$

or

$$s''(t) = \frac{\langle\gamma'(t),\gamma''(t)\rangle}{|\gamma'(t)|^2}$$

I leave it to you to solve for $\gamma''(s)$ and find its norm from the work so far.

Example. A curve of constant curvature and torsion is a helix.

(In fact, the converse is also true, but we leave that for you to prove.)

Suppose $\kappa(s) \equiv \kappa$ and $\tau(s) \equiv \tau$. We want to prove that γ is a helix. Consider the curve

$$\alpha(s) = \gamma''(s) = \kappa N(s).$$

We first prove that $\alpha(s)$ is a circle.

1. $\alpha(s)$ has constant norm κ, so α is on a sphere of radius κ around the origin.

2. We claim α is planar.

We compute

$$\alpha'(s) = \kappa(-\kappa T - \tau B)$$
$$= -\kappa^2 T - \kappa\tau B$$

$$\alpha''(s) = -\kappa^3 N - \kappa\tau^2 N$$
$$= -\kappa(\kappa^2+\tau^2) N$$

$$\alpha'''(s) = +\kappa^2(\kappa^2+\tau^2) T + \kappa\tau(\kappa^2+\tau^2) B$$

$$= -(\kappa^2+\tau^2)\left(-\kappa^2 T - \kappa\tau B\right)$$

$$= -(\kappa^2+\tau^2)\, \alpha'(s).$$

Thus the triple product in numerator of

$$\tau_\alpha(s) = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

must vanish, since α''' is a scalar multiple of α'. We then see that $\tau_\alpha(s) \equiv 0$, so α is planar.

Since $\alpha(s)$ is a circle of radius κ, and $\alpha'(s)$ has norm $\kappa\sqrt{\kappa^2+\tau^2}$, we can write

$$\gamma''(s) = \alpha(s) = \left(\kappa \cos \sqrt{\kappa^2+\tau^2}\, s,\ \kappa \sin \sqrt{\kappa^2+\tau^2}\, s,\ 0 \right)$$

Integrating twice with respect to s, we get

$$\gamma(s) = \left(\frac{-\kappa}{\kappa^2+\tau^2} \cos \sqrt{\kappa^2+\tau^2}\, s,\ \frac{-\kappa}{\kappa^2+\tau^2} \sin \sqrt{\kappa^2+\tau^2}\, s,\ Cs + D \right)$$

We can take $D = 0$ by choosing coordinates so that $\gamma(0) = \left(\frac{-\kappa}{\kappa^2+\tau^2}, 0, 0\right)$. To solve for C, we observe that

$$1 = |\gamma'(s)|^2 = \frac{\kappa^2}{\kappa^2+\tau^2} + \frac{\kappa^2}{\kappa^2+\tau^2} + C^2, \text{ or } C^2 = \tau^2 - \kappa^2.$$

With this example in hand, it may seem that constant κ + constant τ is very restrictive, since we should get some restrictions from assuming that

$$\kappa \equiv C \quad \text{or} \quad \tau \equiv C.$$

Not so!

Theorem [Ghomi, 2006] If γ is a curve of maximum curvature K and $K_2 \geq K$ then there is a curve γ_2 of constant curvature K_2 for any given $\epsilon > 0$ so that

$$|\gamma - \gamma_2| < \epsilon \quad \text{and} \quad |\gamma' - \gamma_2'| < \epsilon.$$

A similar statement holds for curves of constant torsion. Further, both theorems provide <u>closed</u> approximations to closed initial curves γ.