Differential Geometry - Lecture 2

Last time we ended by defining a <u>regular</u> curve

$$\alpha(s) : (a,b) \to \mathbb{R}^3$$

so that $\alpha'(s) \neq \vec{0}$.

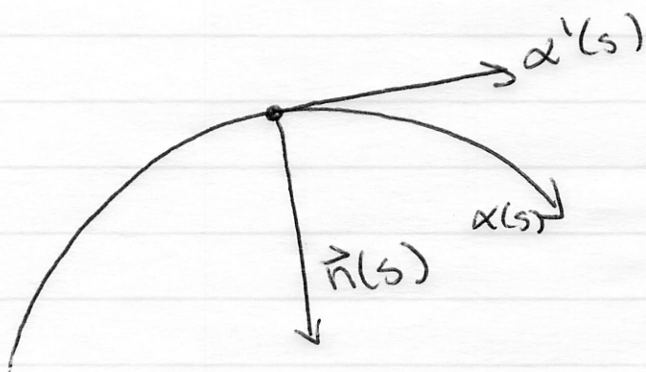
And the curvature $|\alpha''(s)| = \kappa(s)$ and normal vector $\vec{n}(s)$ of a curve

$$\alpha''(s) = \kappa(s)\,\vec{n}(s).$$

Observe that $\vec{n}(s) \cdot \alpha'(s) \equiv 0$:

$$\frac{d}{ds}\left(\alpha'(s) \cdot \alpha'(s)\right) = \frac{d}{ds} 1 = 0.$$

$$\begin{gathered} \| \\ 2\,\alpha''(s) \cdot \alpha'(s) = 2\kappa(s)\left(\vec{n}(s) \cdot \alpha'(s)\right). \end{gathered}$$

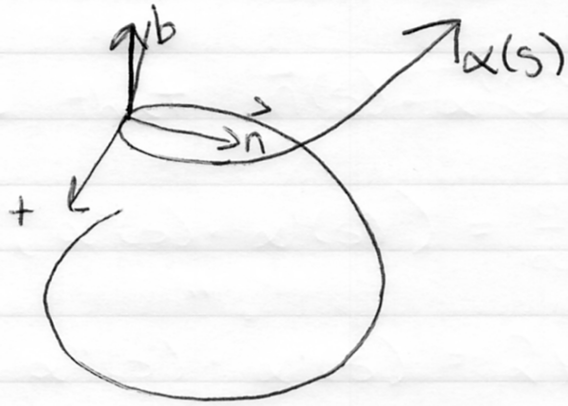


Definition. The plane determined by $\alpha'(s)$ and $\vec{n}(s)$ is called the <u>osculating plane</u> at s.

We will now restrict our attention to curves where $\alpha''(s)$ does not vanish, so this plane is well-defined.

We denote $\alpha'(s)$ by $\vec{t}(s)$. Then we define the <u>binormal</u> $\vec{b}(s)$ by

$$\vec{b}(s) = \vec{t}(s) \times \vec{n}(s).$$



The derivative of the binormal measures the rate at which $\vec{\alpha}(s)$ is leaving the osculating plane.

Let's compute $\vec{b}'(s)$:

$$\vec{b}'(s) = \frac{d}{ds}\, \vec{t}(s) \times \vec{n}(s)$$

$$= \vec{t}'(s) \times \vec{n}(s) + \vec{t}(s) \times \vec{n}'(s)$$

$$= \vec{t}(s) \times \vec{n}'(s).$$

In particular,

$$\vec{b}'(s) \cdot \vec{b}(s) = 0 \quad (\text{since } \vec{b}(s) \text{ is unit})$$

$$\vec{b}'(s) \cdot \vec{t}(s) = 0 \quad (\text{by above})$$

so $\vec{b}'(s)$ must be a scalar multiple of $\vec{n}(s)$: we define $\tau(s)$ by

$$\vec{b}'(s) = \tau(s)\, \vec{n}(s).$$

Definition. The number $\tau(s)$ is called the <u>torsion</u> of α at s.

We note that plane curves have zero torsion, and that any curve with nonvanishing curvature and vanishing torsion is a plane curve.

We call $\vec{t}(s), \vec{n}(s), \vec{b}(s)$ the <u>Frenet</u> <u>frame</u> on $\alpha(s)$. We know

$$t'(s) = \kappa(s)\,\vec{n}(s)$$

$$\begin{aligned} n'(s) &= \frac{d}{ds}\left(\vec{b}(s) \times \vec{t}(s)\right) \\ &= \vec{b}'(s) \times \vec{t}(s) + \vec{b}(s) \times t'(s) \\ &= \tau(s)\,\vec{n}(s) \times \vec{t}(s) + \vec{b}(s) \times \kappa(s)\,\vec{n}(s) \\ &= -\tau(s)\,\vec{b}(s) - \kappa(s)\,\vec{t}(s). \end{aligned}$$

$$b'(s) = \tau(s)\,\vec{n}(s)$$

These are the <u>Frenet</u> <u>formulas</u>. (We'll use these later.)

It seems like bending $(\kappa(s))$ and twisting $(\tau(s))$ encapsulate all the possible deformations of a space curve.

This fact is expressed by

start here

Fundamental Theorem of Local Theory of Curves: Given differentiable functions $\kappa(s) > 0$ and $\tau(s)$ on I, there exists a regular parametrized curve $\alpha : I \to \mathbb{R}^3$ so that

s is the arclength of α
$\kappa(s)$ is the curvature of α
$\tau(s)$ is the torsion of α

Further, any other curve $\bar{\alpha}(s)$ with the same curvature and torsion differs from $\alpha(s)$ by a rigid motion.

Another way of saying this; usually we use three functions $x(s), y(s), z(s)$ to specify a curve. If $\alpha(s)$ is parametrized by arclength, then

$$x'^2(s) + y'^2(s) + z'^2(s) = 1$$

so two "should" suffice (we could integrate up from $z'(s)$ to $z(s)$).

We won't prove existence (that's the theory of ODE's) but we <u>will</u> prove uniqueness.

Proof (of part 2).

Note that arclength, curvature, and torsion are all invariant under rigid motions.

So suppose $\alpha(s)$ and $\bar{\alpha}(s)$ have the same $\kappa(s)$ and $\tau(s)$; we can certainly arrange for

$$\begin{aligned} t(0) &= \bar{t}(0) \qquad\qquad \alpha(0) = \bar{\alpha}(0) \\ n(0) &= \bar{n}(0) \\ b(0) &= \bar{b}(0) \end{aligned}$$

by a rigid motion of $\bar{\alpha}(s)$.

Now let's consider a kind of "total distance" between the frames $t(s), n(s), b(s)$ and $\bar{t}(s), \bar{n}(s), \bar{b}(s)$:

We take

$$d(s) = |t(s) - \bar{t}(s)|^2 + |n(s) - \bar{n}(s)|^2 + |b(s) - \bar{b}'(s)|^2.$$

Then

$$\begin{aligned} d'(s) = &\frac{d}{ds}(t(s) - \bar{t}(s)) \cdot (t(s) - \bar{t}(s)) \\ &+ \frac{d}{ds}(n(s) - \bar{n}(s)) \cdot (n(s) - \bar{n}(s)) \\ &+ \frac{d}{ds}(b(s) - \bar{b}(s)) \cdot (b(s) - \bar{b}(s)) \end{aligned}$$

$$\begin{aligned} = &2\,(t'(s) - \bar{t}'(s)) \cdot (t(s) - \bar{t}^*(s)) \\ &+ 2\,(n'(s) - \bar{n}'(s)) \cdot (n(s) - \bar{n}(s)) \\ &+ 2\,(b'(s) - \bar{b}'(s)) \cdot (b(s) - b(s)) \end{aligned}$$

Now using the Frenet equations:

$$\begin{aligned} = &2\kappa(s)\,(n(s) - \bar{n}(s)) \cdot (t(s) - \bar{t}(s)) \\ &- 2\kappa(s)\,(t(s) - \bar{t}(s)) \cdot (n(s) - \bar{n}(s)) \\ &- 2\tau(s)\,(b(s) - \bar{b}(s)) \cdot (n(s) - \bar{n}(s)) \\ &+ 2\tau(s)\,(n(s) - \bar{n}(s)) \cdot (b(s) - \bar{b}(s)) \end{aligned}$$

$$= 0.$$

Thus $d'(s) \equiv 0$ and so, since $d(0) = 0$, $d(s) \equiv 0$. This means that $t(s), n(s), b(s) = \bar{t}(s), \bar{n}(s), \bar{b}(s)$ everywhere.

In particular, since

$$\alpha'(s) = t(s) = \bar{t}(s) = \bar{\alpha}'(s)$$

we have

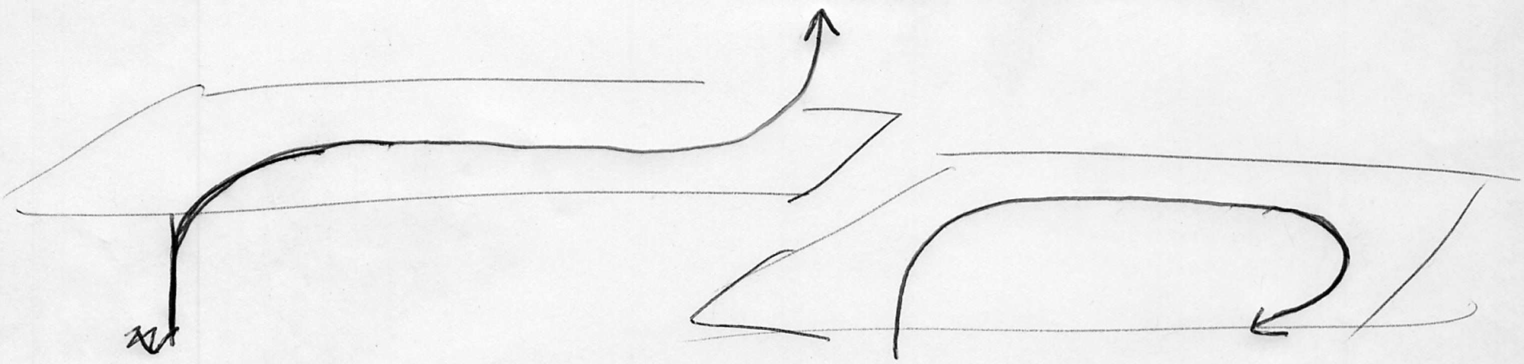
$$\frac{d}{ds}\left(\alpha(s) - \bar{\alpha}(s)\right) \equiv \vec{0}.$$

But this means

$$\alpha(s) = \bar{\alpha}(s) + \vec{c}$$

for some constant vector $\vec{c}$, and since $\alpha(0) = \bar{\alpha}(s)$ by assumption, we must have $\alpha = \bar{\alpha}$, completing the proof.

Notice that we used the Frenet frame (and the assumption that $\kappa(s) \neq 0$) in a really nontrivial way: if we relaxed that assumption we'd be faced with examples like:



(and our theorem would be false!)

$\longrightarrow$ Prove $\tau \equiv 0 \iff$ planar, $\kappa \equiv 0 \iff$ linear

Day 3

Second remark. We observe that given any regular ~~plane~~ curve $\alpha : I \to \mathbb{R}^3$ we can find $\beta : J \to \mathbb{R}^3$ parametrized by arclength with the same trace as α.

If we let

$$s(t) = \int_{t_0}^{t} |\alpha'(t)|\,dt$$

then since $s'(t) = |\alpha'(t)| \neq 0$, this function has a differentiable inverse "$F(s)$" by the inverse function theorem.

↳ state Inverse Fn Theorem

Further, since "$F(s(t))$" $= t$, we have

$$\text{"}F'(s(t))\text{"} \cdot s'(t) = 1.$$

So

$$\frac{d}{ds}\alpha(F(s)) = \alpha'(F(s)) \cdot F'(s)$$

and

$$\left|\frac{d}{ds}\alpha(F(s))\right| = |\alpha'(F(s))|\, F'(s)$$

$$= s'(F) \cdot F'(s)$$

$$= 1.$$

Thus $\alpha(t(s)) = \beta(s)$ is an arclength parametrized curve with the same trace as α.

We now define:

Definition. If $\alpha(t)$ is any regular parametrized curve (that is, $|\alpha'(t)| \neq 0$) we define <u>curvature</u> and <u>torsion</u> for $\alpha(t)$ at t by saying

$\kappa(t)$ = the curvature of the reparametrization of α by arclength at t

$\tau(t)$ = the torsion of the reparametrization of α by arclength at t.

Note: The fundamental theorem still works!