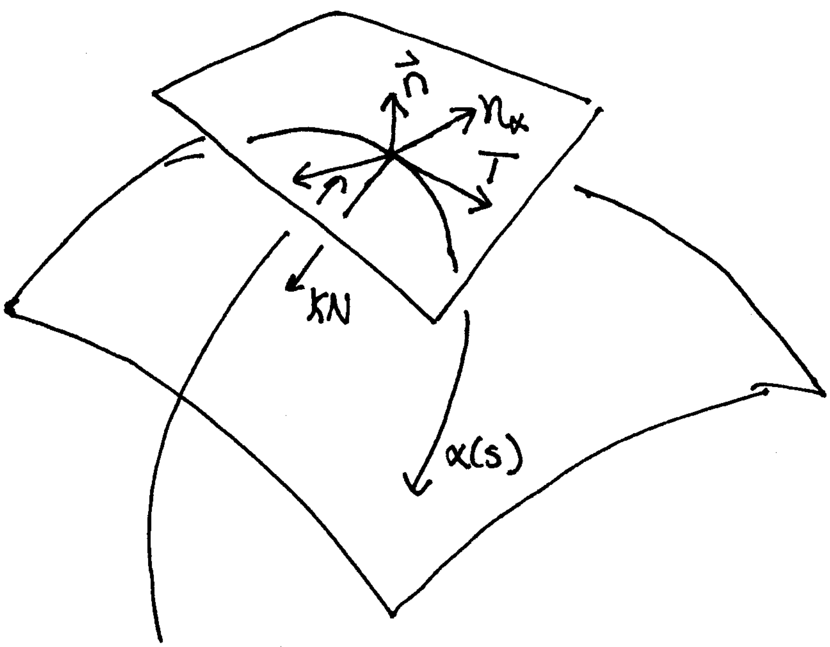
Geodesic Curvature

We have previously shown that the projection of the normal of a curve $\alpha(s)$ to the tangent plane is given by



$$\alpha''(s) - \langle \vec{n}, \alpha''(s) \rangle = \left(u'' + (u')^2 \Gamma_{11}^1 + 2u'v' \Gamma_{12}^1 + (v')^2 \Gamma_{22}^1\right) X_u$$
$$+ \left(v'' + (u')^2 \Gamma_{11}^2 + 2u'v' \Gamma_{12}^2 + (v')^2 \Gamma_{22}^2\right) X_v$$

We now ~~make~~ introduce ~~a~~ ~~this~~ related vector: the <u>intrinsic normal</u> n_α of $\alpha \subset S$. ~~The length of geodesic curvature is given by~~

$$\cancel{k_g = \langle \quad , n_\alpha(s) \rangle_{I_p}}$$

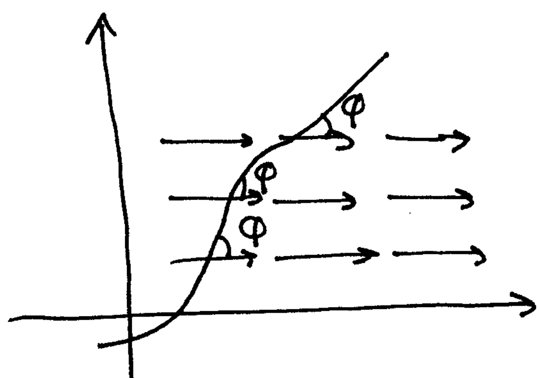
The intrinsic normal n_α is a unit vector in $T_\alpha S$ chosen to be $\perp$ to $\alpha'(s)$, and so that

$$\alpha' \times n_\alpha = \text{the surface normal}$$

If $\alpha(s) = X(u(s), v(s))$, then the signed geodesic curvature of α is given by the formula

$$K_g(s) = \left\langle \begin{bmatrix} u'' \\ v'' \end{bmatrix}, \eta_\alpha \right\rangle_{I_{\alpha(s)}}.$$

We will now use this to prove a nice intrinsic formula for K_g. Suppose $X(u,v)$ has $F=0$, and define



$$\varphi(s) = \angle \alpha'(s), X_u(\alpha(s))$$

Proposition. $K_g(s) = \dfrac{1}{2\sqrt{EG}} \left[G_u v' - E_v u' \right] + \varphi'(s).$

Proof. Since $E = \langle x_u, x_u \rangle = |x_u|^2$, and $G = \langle x_v, x_v \rangle = |x_v|^2$, the vectors

$$e_u = \frac{x_u}{\sqrt{E}} \quad \text{and} \quad e_v = \frac{x_v}{\sqrt{G}}$$

are unit vectors in the coordinate directions. Since these are $\perp$ (F is 0), we have

$$\alpha'(s) = \cos \varphi_{\xi}(s)\, e_u + \sin \varphi_{\xi}(s)\, e_v$$

$$n_\alpha(s) = -\sin \varphi_{\xi}(s)\, e_u + \cos \varphi_{\xi}(s)\, e_v$$

Now

$$\begin{aligned} \alpha''(s) &= (-\sin \varphi_{\xi})(\varphi')\, e_u + \cos \varphi\, (e_u)' \\ &\quad (\cos \varphi)(\varphi')\, e_v + \sin \varphi\, (e_v)'. \\ &= \varphi'\, n_\alpha + \cos \varphi\, (e_u)' + \sin \varphi\, (e_v)'. \end{aligned}$$

Now $K_g = \langle \alpha'', n_\alpha \rangle$, so we get

$$K_g = \varphi' + \cos\varphi \langle e_u', n_\alpha \rangle + \sin\varphi \langle e_v', n_\alpha \rangle$$

$$= \varphi' + \cos\varphi(-\sin\varphi)\langle e_u', e_u\rangle + \cos\varphi\cos\varphi\langle e_u', e_v\rangle$$
$$+ \sin\varphi(-\sin\varphi)\langle e_v', e_u\rangle + \sin\varphi\cos\varphi\langle e_v', e_v\rangle.$$

But e_u and e_v are <u>unit</u> vectors, so $\langle e_u, e_u'\rangle$ and $\langle e_v, e_v'\rangle$ are both zero. Further $\langle e_u, e_v\rangle = 0$, so

$$\frac{d}{ds}\langle e_u, e_v\rangle = \langle e_u', e_v\rangle + \langle e_u, e_v'\rangle = 0.$$

Thus

$$K_g = \varphi' + \langle e_u', e_v\rangle. \qquad \text{(Pretty neat, huh?)}$$

Now we must compute $e_u' = \dfrac{d}{ds}\dfrac{X_u(u(s),v(s))}{\sqrt{E(u(s),v(s))}}$.

Using the chain rule, this is

$$e_u' = \frac{1}{\sqrt{E}}\left(x_{uu}\, u' + x_{uv}\, v' \right)$$

$$+ -\frac{1}{2}\frac{1}{E^{3/2}}\left(E_u\, u' + E_v\, v' \right) x_u$$

Now $e_v = \frac{x_v}{\sqrt{G}}$, so we compute

$$\langle x_{uu}, x_v \rangle = \frac{\partial}{\partial u}\langle x_u, x_v \rangle - \langle x_u, x_{vu} \rangle$$

$$= F_u - \frac{1}{2} E_v$$

But $F = 0$, so $F_u = 0$ and this is

$$\langle x_{uu}, x_v \rangle = -\frac{1}{2} E_v$$

Similarly,

$$\langle x_{uv}, x_v \rangle = \langle x_{vu}, x_v \rangle = \frac{1}{2} G_u.$$

Last, $\langle x_u, e_v \rangle = 0$, since e_v is in the direction of x_v and $x_u \perp x_v$.

So now we have

$$\langle e'_u, e_v \rangle = \frac{1}{\sqrt{EG}}\left(-\frac{1}{2}E_v u' + \frac{1}{2}G_u v'\right),$$

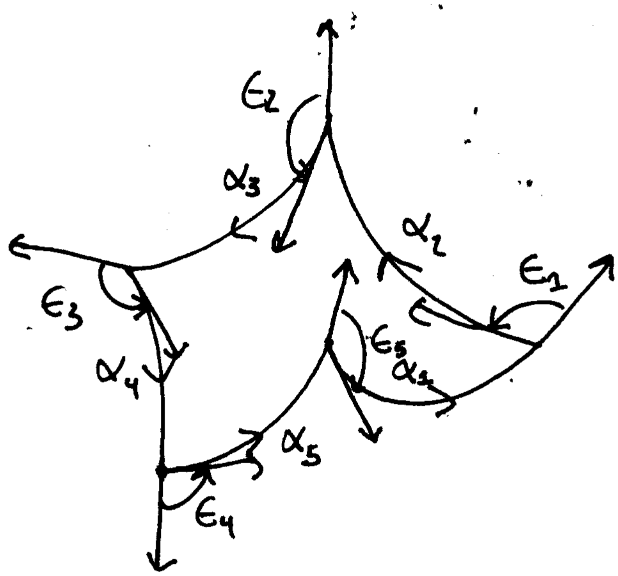
which is exactly what we wanted to prove.

Now we have given as a homework assignment the formula

$$K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$$

when $F = 0$.

We will now combine these formulae to yield a triumphant moment in the intrinsic geometry of surfaces.



Suppose $\alpha_1, \dots, \alpha_n$ are differentiable curves which bound a region R in S.
We assume that

i) $R = x(P)$ for some open set P in the u-v plane which is simply connected (no holes).

ii) We have $F=0$ on an open set containing the α_i and R.

At each corner we define a (signed) turning angle ϵ_i in the expected way. (Note that ϵ_5 above is <u>negative</u>.)

We can also define

$$\varphi_i(s) = \angle\, x_u, \alpha_i'$$

along each α_i. We then have

Hopf's Umlaufsatz. The total turning of the tangent vector around α is 2π.
Or

$$\sum_i \int \varphi_i'(s)\,ds + \sum_i \epsilon_i = 2\pi.$$

(This is actually hard to prove, and was published only in 1938.)

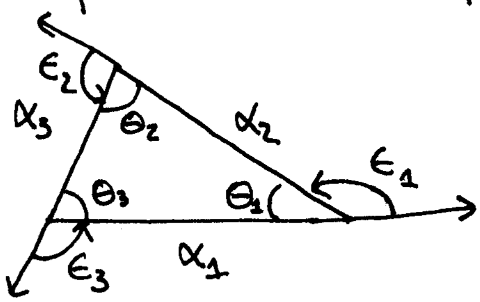
Using this, we show the

Gauss-Bonnet Theorem (1828-1848)

$$\sum_i \int_{\alpha_i} K_g(s)\,ds + \iint_R K\, dArea = 2\pi - \sum_i \epsilon_i$$

The statement of this theorem is so shocking that we must parse a few examples before we can prove it.

Example. In the plane, take a triangle.



$K_g = 0$ on all sides. $K = 0$ everywhere.

So we read

$$0 = 2\pi - \sum \epsilon_i.$$

or

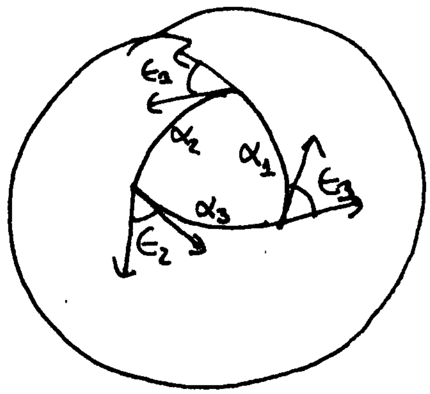
$$0 = 2\pi - \cancel{\sum} (\pi - \theta_1) - (\pi - \theta_2) - (\pi - \theta_3)$$

or

$$\pi = \theta_1 + \theta_2 + \theta_3.$$

(A cool proof of a fact you know!)

Example. On the sphere, take a geodesic triangle



Again, $k_g = 0$ on all sides. But now $K \equiv 1$, so we read

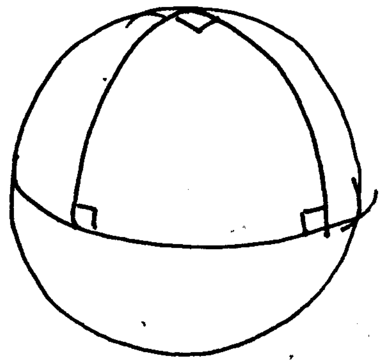
$$\iint 1 \, dArea = 2\pi - \sum \epsilon_i$$
$$= (\theta_1 + \theta_2 + \theta_3) - \pi.$$

This is a shocker:

1) Spherical triangles have angle sum <u>greater than π</u>!

2) The "extra" angle is equal to the <u>area of the triangle</u>!!

Observe that



angle sum = $\frac{3\pi}{2}$

area = $\frac{1}{8}$ of sphere = $\frac{4\pi}{8} = \frac{1}{2}\pi$.

~~Example~~

Now let's prove it. We know

$$\sum_i \int_{\alpha_i} K_g(s)\,ds = \sum_i \int_{\alpha_i} \frac{1}{2\sqrt{EG}}\left[G_u v' - E_v u'\right] + \int_{\alpha_i} \varphi_i'(s)\,ds$$

Using the Umlaufsatz, we see that

$$\sum_i \int \varphi_i'(s)\,ds = 2\pi - \sum \epsilon_i.$$

Now the α_i form the boundary of R. And we know for any smooth functions P, Q on $\mathbb{R}^2$ we have

$$\int_{\partial R} Pu' + Qv'\,ds = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\,dv,$$

by Green's theorem.

In our case

$$P = \frac{-E_v}{\sqrt{EG}}, \qquad Q = \frac{G_u}{\sqrt{EG}}$$

and

$$\sum \int_{\alpha_i} \frac{1}{2\sqrt{EG}} \left[G_u v' - E_v u' \right] = \frac{1}{2} \iint_R \left(\frac{G_u}{\sqrt{EG}} \right)_u + \left(\frac{E_v}{\sqrt{EG}} \right)_v du\,dv.$$

Now $d\text{Area} = \sqrt{EG - F^2}\, du\,dv$, so we have

$$= \iint \frac{1}{2\sqrt{EG}} \left(\left(\frac{G_u}{\sqrt{EG}} \right)_u + \left(\frac{E_v}{\sqrt{EG}} \right)_v \right) d\text{Area}$$

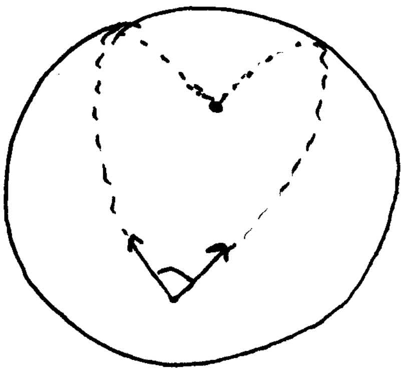
$$= - \iint K \, d\text{Area}.$$

So

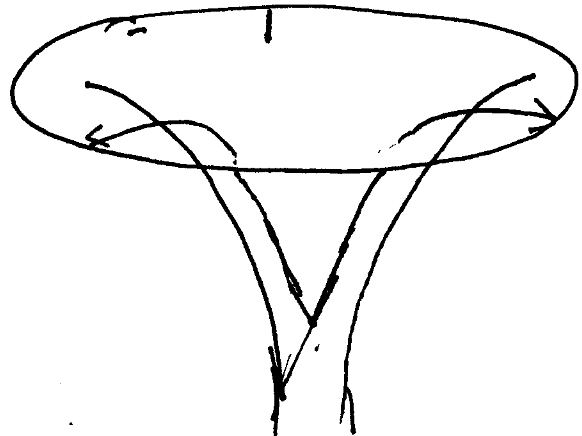
$$\sum \int_{\alpha_i} K_g(s)\, ds + \iint_R K \, d\text{Area} = 2\pi - \epsilon_i,$$

as desired!

We now explain one last cool thing: Consider two geodesics leaving a point

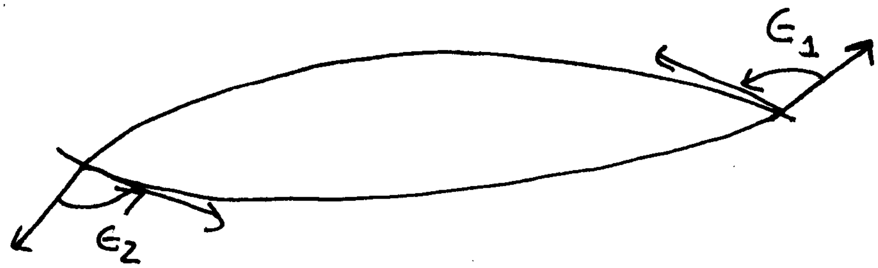


on the sphere, they bend towards each other and meet again



on a surface of negative curvature, they diverge, or do not bound a simply connected region

Proof. Suppose two geodesics met twice on a surface of negative curvature



If the region between them is simply connected

$$\iint_R K = 2\pi - \epsilon_1 - \epsilon_2 .$$

But $\epsilon_1, \epsilon_2 < \pi$, so this is > 0. But $\iint_R K < 0$. $\times\!\!\times$