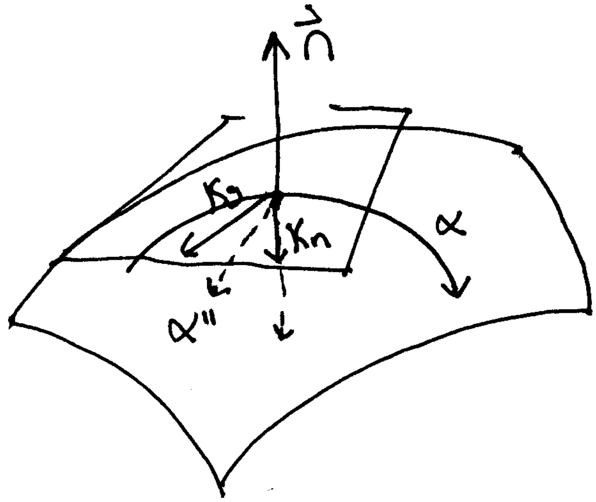
# Theory of Geodesics – Intrinsic and Extrinsic.

We are now ready to consider the "straight lines" of differential geometry. We will do this first from the extrinsic point of view and then from the intrinsic.




Definition. Given a curve $\alpha(s)$ in a surface $S$, we have

$$\alpha''(s) = k_n \vec{N} + k_g \vec{V}$$

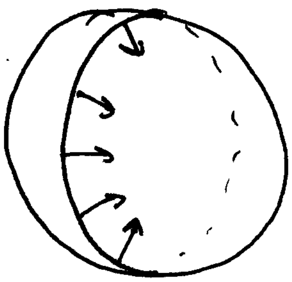
where $\vec{n}$ is the normal to $S$ and $\vec{V}$ is tangent to $S$.

$k_n$ is the <u>normal</u> curvature of $\alpha$

$k_g$ is the <u>geodesic</u> curvature of $\alpha$.

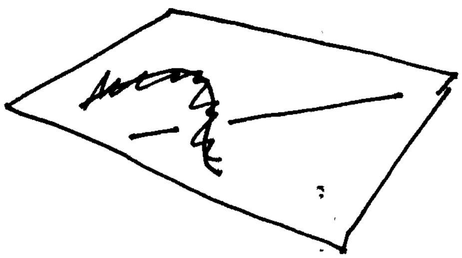
Definition. $\alpha(s)$ is a <u>geodesic</u> if $\kappa_g(s) \equiv 0$.

Examples.



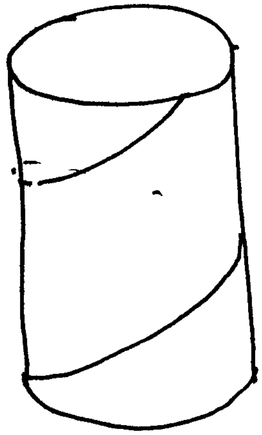
$S$ a sphere, $\alpha(s)$ a great circle.

why? $\alpha$ is a normal section of $S$



$S$ a plane, $\alpha(s)$ a straight line

why? $\alpha''(s) = 0$



$S$ a cylinder, $\alpha(s)$ a helix

why? more complicated.

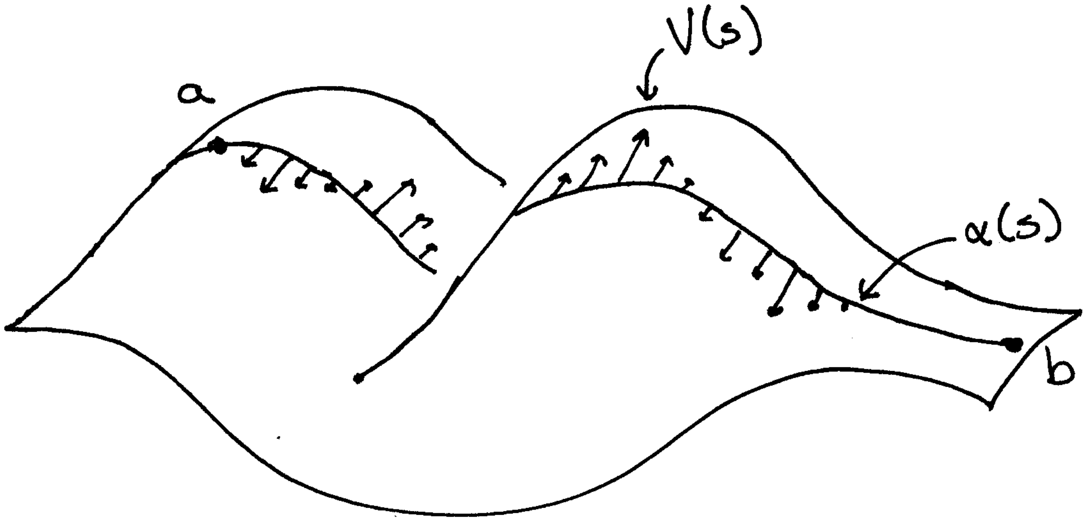
$$\alpha(s) = (\cos(\kappa s), \sin(\kappa s), \ell s)$$

where $\kappa^2 + \ell^2 = 1$. So

$$\alpha'(s) = (-\kappa \cancel{\cos} \sin \kappa s, \kappa \cos \kappa s, \ell)$$

$$\alpha''(s) = (-\kappa^2 \cos \kappa s, -\kappa^2 \sin \kappa s, 0),$$ which is normal to $S$.

We call geodesics "lines" because they are locally critical for length in the following sense.




Definition. If $V(s)$ is a differentiable map from $\mathbb{R}$ to $\mathbb{R}^3$ so that $V(s) \in T_{\alpha(s)} S$ for all $s$, then $V$ is a <u>vector field on $\alpha(s)$</u>.

The <u>variation of length of $\alpha$</u> under $V$ is

$$\frac{d}{d\epsilon} \text{Length}\left(\alpha(s) + \epsilon V(s)\right)\Big|_{\epsilon=0} = D_{\vec{V}} \text{Len}(\alpha).$$

Proposition. We have
$D_V \text{Len}(\alpha) = 0$ for all vector fields $V$ on $\alpha$ with $\vec{v}(0) = \vec{v}(1) = \vec{0}$ $\iff$ $\alpha$ is a geodesic joining $\alpha(0) = a$ and $\alpha(1) = b$.

Proof. We will prove this using integration by parts.

$$\mathrm{Len}\big(\alpha(s)+\epsilon v(s)\big) = \int_0^1 |\alpha'(s)+\epsilon v'(s)|\, ds$$

Differentiating under the integral,

$$^{*}\, D_{\vec{v}}\,\mathrm{Len}(\alpha) = \int_0^1 \frac{d}{d\epsilon}\, |\alpha'(s)+\epsilon v'(s)| \Big|_{\epsilon=0} ds$$

$$= \int_0^1 \cancel{\tfrac{1}{2}} \langle \alpha'(s)+\cancel{\epsilon v'(s)}, \alpha'(s)+\cancel{\epsilon v'(s)} \rangle^{-1/2} \cdot \cancel{2} \langle \alpha'(s)+\cancel{\epsilon v'(s)}, v'(s)\rangle \Big|_{\epsilon=0} ds$$

$$= \int_0^1 \langle \alpha'(s), v'(s)\rangle\, ds$$

Now we integrate by parts, observing

$$\frac{d}{ds}\langle \alpha'(s), v(s)\rangle = \langle \alpha''(s), v(s)\rangle + \langle \alpha'(s), v'(s)\rangle$$

so

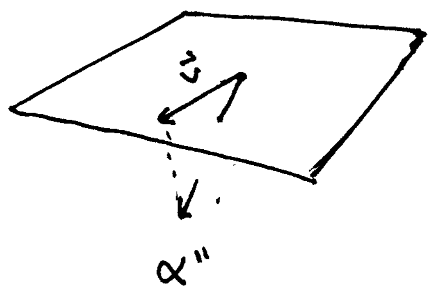
$$D_{\vec{v}}\,\mathrm{Len}(\alpha) = \langle \alpha'(s), v(s)\rangle \Big|_0^1 - \int_0^1 \langle \alpha''(s), v(s)\rangle\, ds.$$

" $\Rightarrow$ "

Suppose this is zero for all, $V$ with $v(0)=v(1)=0$.
Then for any such $\vec{v}$,

$$\int_0^1 \langle \alpha''(s), \vec{v}(s) \rangle = 0.$$

We claim that means $\alpha''(s)$ is normal to $T_{\alpha(s)}S$ for all $s$. If not, choose $\vec{v}(s)$ to agree with the projection of $\alpha''$ to $T_{\alpha(s)}S$ near $s$, and let $\vec{v}(s)=0$ everywhere else. For such a $v$,




$$\int_0^1 \langle \alpha''(s), \vec{v}(s) \rangle \, ds > 0,$$ a contradiction.

" $\Leftarrow$ " If $\alpha$ is a geodesic, $\alpha''$ is normal to $T_{\alpha(s)}S$. But $\vec{v}(s) \in T_{\alpha(s)}S$, so $\langle \alpha''(s), V(s) \rangle = 0$ for all $s$. Thus $D_{\vec{v}} \operatorname{Len} \alpha = 0$, as desired.

How can we compute geodesic curvature in <u>intrinsic</u> terms?

Suppose

$$\alpha'(s) = u'(s)\, X_u + v'(s)\, X_v .$$

Then

$$\alpha''(s) = \cancel{u''(s) X_u} + \cancel{u'(s) X_{uu}} + \cancel{v''(s) X}$$

$$= u''(s)\, X_u + u'(s)\left[\frac{d}{ds} X_u\right] + v''(s)\, X_v + \left[\frac{d}{ds} X_v\right] v'(s).$$

Now

$$\frac{d}{ds} X_u(u(s), v(s)) = u'(s)\, X_{uu} + v'(s)\, X_{uv}$$

and

$$\frac{d}{ds} X_v(u(s), v(s)) = u'(s)\, X_{vu} + v'(s)\, X_{vv}$$

The tangential portion of $X_{uu}$, $X_{uv}$, and $X_{vv}$ determine the geodesic curvature of $\alpha$. In particular, we have

$$\alpha''(s) = u'' + (u')^2 \Gamma_{11}^1 + 2u'v' \Gamma_{12}^1 + (v')^2 \Gamma_{22}^1 \quad X_u$$

$$v'' + (u')^2 \Gamma_{11}^2 \; 2u'v' \Gamma_{12}^2 + (v')^2 \Gamma_{22}^2 \quad X_v$$

$$+ \text{(a bunch of stuff) } N.$$

These give us the differential equations of a geodesic:

$$u'' + (u')^2 \Gamma_{11}^1 + 2u'v' \Gamma_{12}^1 + (v')^2 \Gamma_{22}^1 = 0$$

$$v'' + (u')^2 \Gamma_{11}^2 + 2u'v' \Gamma_{12}^2 + (v')^2 \Gamma_{22}^2 = 0.$$

Example. Geodesics on a surface of revolution.

$$X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)).$$

$$\Gamma_{11}^1 = 0. \quad \Gamma_{11}^2 = -\frac{\varphi\varphi'}{(\varphi')^2 + (\psi')^2}. \quad \Gamma_{12}^1 = \frac{\varphi'}{\varphi}$$

$$\Gamma_{12}^2 = 0 \qquad \Gamma_{22}^1 = 0 \qquad \Gamma_{22}^2 = \frac{\varphi'\varphi'' + \psi'\psi''}{(\varphi')^2 + (\psi')^2}$$

So we get for the geodesic equations

$$u'' + 2\frac{\varphi'}{\varphi} u'v' = 0.$$

$$v'' - (u')^2 \frac{\varphi\varphi'}{(\varphi')^2 + (\psi')^2} + \frac{\varphi'\varphi'' + \psi'\psi''}{(\varphi')^2 + (\psi')^2}(v')^2 = 0.$$

Claim. Meridians ($u = \text{const}$, $v = v(s)$) are always geodesics.

Since $u = \text{const}$, $u' = 0$, $u'' = 0$ and the first equation is satisfied. The second equation becomes

$$v'' + \frac{\varphi'\varphi'' + \psi'\psi''}{(\varphi')^2 + (\psi')^2}(v')^2 = 0.$$

Now $v(s)$ is parametrized by arclength, so we know that

$$G(v')^2 = 1$$

But $G = (\varphi')^2 + (\psi')^2$, so this shows

$$(v')^2 = \frac{1}{(\varphi')^2 + (\psi')^2} = \frac{1}{G(u,v)}$$

from the chain rule!

Taking derivatives, we get

$$2v'v'' = \frac{-1}{(\varphi'^2 + \psi'^2)^2} \cdot (2\varphi'\varphi'' + 2\psi'\psi'')\, v'$$

So

$$\cancel{2}v'v'' = -\frac{\cancel{2}(\varphi'\varphi'' + \psi'\psi'')}{(\varphi')^2 + (\psi')^2}(v')^3,$$

using our identity $(v')^2 = \dfrac{1}{(\varphi')^2 + (\psi')^2}$, so

$$v'' = \frac{\varphi'\varphi'' + \psi'\psi''}{(\varphi')^2 + (\psi')^2}(v')^2, \quad \text{as desired.}$$

So every <u>meridian</u> is a geodesic.

Claim. Parallels ($u = u(s)$, $v = \text{const}$) are geodesics $\Leftrightarrow$ $\varphi' = 0$ on the parallel.

Our first equation becomes

$$u'' = 0, \quad \text{so} \quad u' = \text{constant. That's ok.}$$

The second is

$$(u')^2 \frac{\varphi\varphi'}{(\varphi')^2 + (\psi')^2} = 0.$$

So either

$$u' = 0 \quad \text{or} \quad \varphi = 0 \quad \text{or} \quad \varphi' = 0$$

If so, $u'$ is <u>always</u> zero, and $(u(s), c)$ is not unit speed.

Then $S$ would not be regular.

(This is all that's left!)

We have proved claim. What about other ~~equations?~~ curves? Consider

$$u'' + 2\frac{\varphi'}{\varphi}u'v' = 0.$$

or

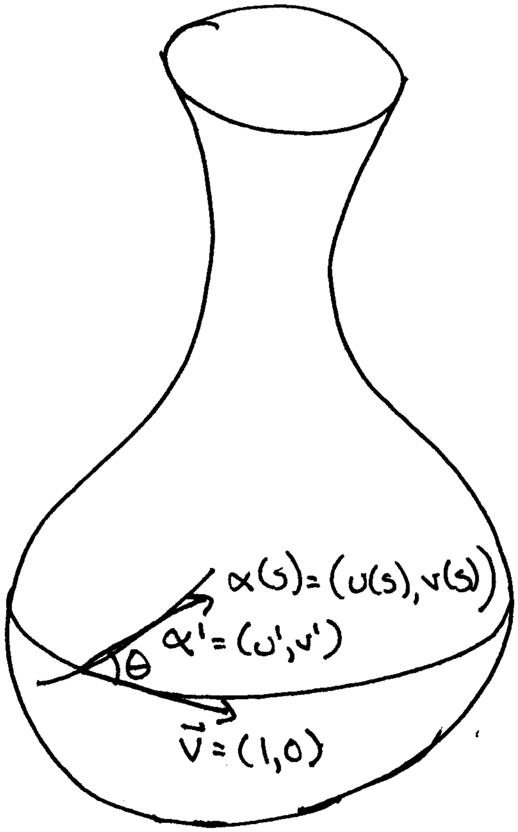
$$\varphi^2 u'' + 2\varphi'\varphi\, u'v' = 0.$$

Now this is

$$\frac{d}{ds}(\varphi^2 u') = \varphi^2 u'' + u' 2\varphi\varphi' v' = 0,$$

or $\varphi^2 u' =$ constant.

Let's interpret this geometrically!




We note that

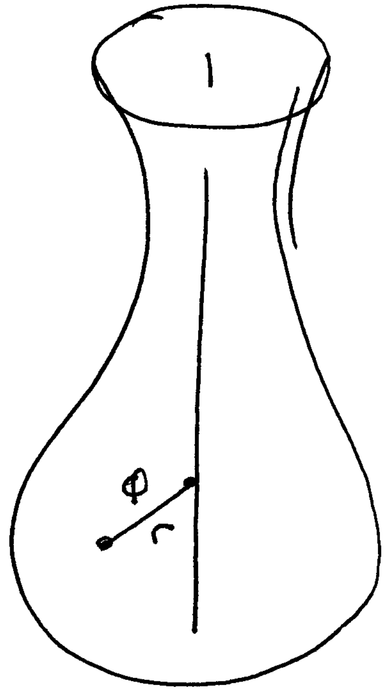
$$\cos\theta = \frac{\langle (u',v'), (1,0) \rangle_{I_p}}{|(u',v')|\,|(1,0)|}$$

Now $|(u',v')| = |\alpha'| = 1$, since $\alpha$ is arclength parametrized. Further, $|(1,0)| = \sqrt{E} = \varphi$, $\langle (u',v'), (1,0) \rangle = \varphi^2 u'$.

So

$$\cos\theta = \varphi u'$$

Thus $\varphi^2 u' = \varphi \cos\theta = \text{constant}.$




This is <u>Clairaut's relation</u> for geodesics:

$$r \cos \theta = \text{const.}$$

Consequences. A geodesic is never tangent to a meridian unless it is a meridian. So all other geodesics spiral clockwise or counterclockwise.

Example. Let $S$ be a paraboloid. Any geodesic not a meridian intersects itself an infinite # of times.