The Gauss Theorem and Mainardi-Codazzi Equations

We know that x_u, x_v, N form a basis for $\mathbb{R}^3$ at p. So we can write

$$
\begin{aligned}
x_{uu} &= \Gamma_{11}^{1} x_u + \Gamma_{11}^{2} x_v + L_1 N \\
x_{uv} &= \Gamma_{12}^{1} x_u + \Gamma_{12}^{2} x_v + L_2 N \\
\| \quad & \qquad \| \qquad\qquad \| \qquad\qquad \| \\
x_{vu} &= \Gamma_{21}^{1} x_u + \Gamma_{21}^{2} x_v + \bar{L}_2 N \\
x_{vv} &= \Gamma_{22}^{1} x_u + \Gamma_{22}^{2} x_v + L_3 N
\end{aligned}
$$

$$
\begin{aligned}
N_u &= a_{11} x_u + a_{21} x_v \\
N_v &= a_{12} x_u + a_{22} x_v
\end{aligned}
$$

Now

$$
\begin{aligned}
\langle x_{uu}, N \rangle &= L_1 = e \\
\langle x_{uv}, N \rangle &= L_2 = \langle x_{vu}, N \rangle = \bar{L}_2 = f \\
\langle x_{vv}, N \rangle &= L_3 = g
\end{aligned}
$$

What about the Γ_{ij}^k? These are called <u>Christoffel</u> <u>symbols</u>, and are defined by

$$\frac{\partial^2}{\partial u_i \partial u_j} X = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial u_k} X + \left(\text{stuff normal to } T_pS\right)$$

In more advanced classes, you'll see this written

$$\frac{\partial^2}{\partial u_i \partial u_j} X = \Gamma_{ij}^{k} \frac{\partial X}{\partial u_k}$$

sum over repeated indices is implied

The notation that we infer a summation over repeated indices is called the <u>Einstein</u> <u>convention</u>, or <u>Einstein</u> <u>summation</u> <u>convention</u>.

(This may be the first time in your life you'll learn a piece of Einstein's mathematics - treasure it!)

Going back to work, we see

$$\frac{1}{2}\frac{\partial}{\partial u}\langle X_u, X_u\rangle = \langle X_u, X_{uu}\rangle$$

$$= \Gamma_{11}^1 \langle X_u, X_u\rangle + \Gamma_{11}^2\langle X_u, X_v\rangle + L_1\langle X_u, N\rangle$$

$$= \Gamma_{11}^1 E + \Gamma_{11}^2 F.$$

Or, we see that

$$\Gamma_{11}^1 E + \Gamma_{11}^2 F = \frac{1}{2} E_u.$$

Now

$$\langle X_{uu}, X_v\rangle = \Gamma_{11}^1 F + \Gamma_{11}^2 G$$

and

$$\langle X_{uu}, X_v\rangle = \frac{\partial}{\partial u}\langle X_u, X_v\rangle - \langle X_u, X_{vu}\rangle$$

$$= F_u - \langle X_u, X_{uv}\rangle$$

$$= F_u - \frac{1}{2}E_v$$

so

$$\Gamma_{11}^1 F + \Gamma_{11}^2 G = F_u - \frac{1}{2}E_v$$

Continuing,

$$\langle X_{uv}, X_u \rangle = \Gamma_{12}^1 E + \Gamma_{12}^2 F = \tfrac{1}{2} E_v$$

$$\langle X_{uv}, X_v \rangle = \Gamma_{12}^1 F + \Gamma_{12}^2 G = \tfrac{1}{2} G_u$$

and

$$\langle X_{vv}, X_u \rangle = \Gamma_{22}^1 E + \Gamma_{22}^2 F = F_v - \tfrac{1}{2} G_u$$

$$\langle X_{vv}, X_v \rangle = \Gamma_{22}^1 F + \Gamma_{22}^2 G = \tfrac{1}{2} G_v$$

We could solve each of these three 2×2 systems of equations for $\Gamma_{11}^i, \Gamma_{12}^i, \Gamma_{22}^i$ since each system is based on the matrix $\begin{bmatrix} E & F \\ F & G \end{bmatrix}$ which has determinant $EG - F^2 \neq 0$.

This wouldn't be hard. Observe

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} E_u \\ F_u - \tfrac{1}{2} E_v \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} \Gamma_{11}^1 \\ \Gamma_{11}^2 \end{bmatrix} = \frac{1}{EG - F^2} \begin{bmatrix} G & -F \\ -F & E \end{bmatrix} \begin{bmatrix} \tfrac{1}{2} E_u \\ F_u - \tfrac{1}{2} E_v \end{bmatrix}$$

So, just to be specific in one case,

$$\Gamma^1_{11} = \frac{1}{EG-F^2}\left(\tfrac{1}{2}GE_u - FF_u + \tfrac{1}{2}FE_v\right),$$

and the other Γ^k_{ij} are similar combinations of E, F, G and their partials.

Proposition. The Christoffel symbols are isometry invariants.

Proof. Isometries preserve the functions E, F, G (and hence their derivatives, etc).

Example. Consider the surface of revolution

$$X(u,v) = (\varphi(v)\cos u, \varphi(v)\sin u, \psi(v)).$$

We recall that

$$E = \varphi^2(v) \qquad F = 0 \qquad G = (\varphi')^2 + (\psi')^2$$

so

$$E_u = 0 \quad E_v = 2\varphi\varphi' \quad F_u = F_v = 0 \quad G_u = 0 \quad G_v = 2\varphi'\varphi + 2\psi'\psi.$$

We then see that

$$\Gamma^1_{11}\,\varphi^2 + \Gamma^2_{11}\,0 = \tfrac{1}{2}\cdot 0$$

$$\Gamma^1_{11}\,0 + \Gamma^2_{11}\left(\varphi'^2 + \psi'^2\right) = 0 - \varphi\varphi'$$

so

$$\Gamma^1_{11} = 0 \qquad \Gamma^2_{11} = \frac{-\varphi\varphi'}{\left(\varphi'^2 + \psi'^2\right)}.$$

And

$$\Gamma^1_{12}\,\varphi^2 + \Gamma^2_{12}\,0 = \frac{1}{2}\varphi\varphi'$$

$$\Gamma^1_{12}\,0 + \Gamma^2_{12}\left(\varphi'^2 + \psi'^2\right) = \tfrac{1}{2}\cdot 0$$

so

$$\Gamma^1_{12} = \frac{\varphi'}{2\varphi}, \qquad \Gamma^2_{12} = 0$$

And then we have

$$\Gamma_{22}^{1}\, \varphi^2 + \Gamma_{22}^{2} \cdot 0 = 0 - \frac{1}{2}\, 0$$

$$\Gamma_{22}^{1}\, 0 + \Gamma_{22}^{2}\left(\varphi'^2 + \psi'^2\right) = \cancel{2}\, \varphi'\varphi + \psi'\psi.$$

so

$$\Gamma_{22}^{1} = 0 \qquad \Gamma_{22}^{2} = \frac{\varphi'\varphi + \psi'\psi}{\left(\varphi'^2 + \psi'^2\right)}$$

Now we would like to express e, f, g in terms of the Γ_{ij}^{k}. We can get started by noting

$$(x_{uu})_v - (x_{uv})_u = 0. \qquad (1)$$

$$(x_{vv})_u - (x_{vu})_v = 0. \qquad (2)$$

$$(N_u)_v - (N_v)_u = 0. \qquad (3)$$

Writing out (1) in the basis x_u, x_v, N we get

$$A_1 x_u + B_1 x_v + C_1 N = 0.$$

for some A_1, B_1, C_1. Of course, x_u, x_v, N are linearly independent, so this is a system of 3 equations

$$A_1 = 0, \quad B_1 = 0, \quad C_1 = 0.$$

Let's work these out. We see that

$$(x_{uu})_v = (\Gamma^1_{11})_v\, x_u + (\Gamma^1_{11})\, x_{uv}$$
$$+ (\Gamma^2_{11})_v\, x_v + (\Gamma^2_{11})\, x_{vv}$$
$$+ e_v N + e N_v$$
$$= (\Gamma^1_{11})_v\, x_u + \left(\Gamma^1_{11}\Gamma^1_{12}\, x_u + \Gamma^1_{11}\Gamma^2_{12} x_v + \Gamma^1_{11} f N\right)$$
$$(\Gamma^2_{11})_v\, x_v + \left(\Gamma^2_{11}\Gamma^1_{22}\, x_u + \Gamma^2_{11}\Gamma^2_{22} x_v + \Gamma^2_{11} g N\right)$$
$$+ e_v N + e a_{12}\, x_u + e a_{22}\, x_v$$

$$(x_{uv})_u = (\Gamma^1_{12})_u\, x_u + (\Gamma^1_{12})\, x_{uu}$$
$$+ (\Gamma^2_{12})_u\, x_v + (\Gamma^2_{12})\, x_{vu}$$
$$+ f_u N + f N_u$$

$$= (\Gamma^1_{12})_u X_u + \Gamma^1_{12}\Gamma^1_{11} X_u + \Gamma^1_{12}\Gamma^2_{11} X_v + \Gamma^1_{12}\, e N$$

$$+ (\Gamma^2_{12})_u X_v + \Gamma^2_{12}\Gamma^1_{21} X_u + \Gamma^2_{12}\Gamma^2_{21} X_v + \Gamma^2_{12}\, f N$$

$$+ f_u N + f a_{11} X_u + f a_{21} X_v$$

If we isolate B_1, the coefficient of X_v, in all this, we get

$$\Gamma^1_{11}\Gamma^2_{12} + (\Gamma^2_{11})_v + \Gamma^2_{11}\Gamma^2_{22} + e a_{22} =$$

$$\Gamma^1_{12}\Gamma^2_{11} + (\Gamma^2_{12})_u + \Gamma^2_{12}\Gamma^2_{21} + f a_{21}.$$

(Gauss formula)

This implies

$$e a_{22} - f a_{21} = \text{a bunch of Christoffel symbols.}$$

But by the Weingarten equations,

$$a_{22} = \frac{fF - gE}{EG - F^2}, \qquad a_{21} = \frac{eF - fE}{EG - F^2}$$

so the lhs is

$$\frac{ef\cancel{F} - egE - \cancel{fe}F + f^2 E}{EG - F^2} = -E\,\frac{eg - f^2}{EG - F^2} = -EK.$$

We have proved that

Theorem. Gauss curvature is invariant under isometries.

Pause to celebrate! Gauss called this the "Theorema Egregium" because it was so darn hard to prove.

Consequence: The helicoid and catenoid have the same Gauss curvature at corresponding points. (!)

(This is <u>not</u> obvious - why should they have the same product of principal curvatures when we can't see why they would ~~even~~ have the same p.c's to start?)

Wrapping up, we observe that repeating this procedure for A_1, the coefficient of x_u gives us an expression for FK.

For the coefficient of N, called C_1, we get

$$\Gamma_{11}^1 f + \Gamma_{11}^2 g + e_v = \Gamma_{12}^1 e + \Gamma_{12}^2 f + f_u$$

or

$$e_v - f_u = e\Gamma_{12}^1 + f(\Gamma_{12}^2 - \Gamma_{11}^1) - g\Gamma_{11}^2. \quad (*)$$

If we repeat the process for (2), we get another expression for FK, an expression for GK and

$$f_v - g_u = e\Gamma_{22}^1 + f(\Gamma_{22}^2 - \Gamma_{12}^1) - g\Gamma_{12}^2 \quad (*)$$

These equations (*) are called

Mainardi-Codazzi equations.

Now recall that the Γ_{ij}^{k} are really expressed in terms of E, F, G, so the Gauss ~~eqn~~ formula and the M-C equations are <u>really</u> eqns relating E, F, G and e, f, g.

Are these the <u>only</u> such relations?

Yes. (Bonnet's theorem).