Computations with II_p: How do we write all this in local coords?

We recall that dN_p maps $x_u \mapsto N_u$ and $x_v \mapsto N_v$. We want to write down dN_p as a matrix in the $\{x_u, x_v\}$ basis. For some $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have

$$N_u = a_{11} x_u + a_{21} x_v$$

$$N_v = a_{12} x_u + a_{22} x_v$$

or (by linearity), if $W = \omega_1 x_u + \omega_2 x_v$,

$$dN \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}.$$

Let's look at dN_p another way: we know that

$$\mathrm{II}_p(\omega) = -\langle \omega_1 x_u + \omega_2 x_v, \omega_1 N_u + \omega_2 N_v \rangle_{I_p}$$
$$= (-\langle x_u, N_u \rangle)\, \omega_1^2 + 2\, \omega_1 \omega_2 (-\langle x_u, N_v \rangle = -\langle x_v, N_u \rangle)$$
$$+ (-\langle x_v, N_v \rangle)\, \omega_2^2$$

~~Also~~ (we have to remember that we proved that $\langle x_u, N_v \rangle = \langle x_v, N_u \rangle$ earlier when we showed that II_p was a quadratic form).

These inner products are named

$$e = -\langle N_u, x_u \rangle_{I_p}$$

$$f = -\langle N_u, x_v \rangle_{I_p} = -\langle N_v, x_u \rangle_{I_p}$$

$$g = -\langle N_v, x_v \rangle_{I_p}$$

Notice that since $\langle N, x_u \rangle = \langle N, x_v \rangle = 0$, these are also given by

$$e = \langle N, x_{uu} \rangle_{I_p}$$

$$f = \langle N, x_{vu} \rangle_{I_p} = \langle N, x_{uv} \rangle_{I_p}$$

$$g = \langle N, x_{vv} \rangle_{I_p}$$

We recall that I_p was given in terms of

$$E = \langle x_u, x_u \rangle_{I_p}$$

$$F = \langle x_u, x_v \rangle_{I_p}$$

$$G = \langle x_v, x_v \rangle_{I_p}$$

So now we can compute

$$\begin{aligned} -f = \langle N_u, x_v \rangle &= \langle a_{11} x_u + a_{21} x_v , x_v \rangle_{I_p} \\ &= a_{11} \langle x_u, x_v \rangle_{I_p} + a_{21} \langle x_v, x_v \rangle_{I_p} \\ &= a_{11} F + a_{21} G \end{aligned}$$

$$-f = \langle N_v, x_u \rangle = a_{12} E + a_{22} F$$

$$-e = \langle N_u, x_u \rangle = a_{11} E + a_{21} F$$

$$-g = \langle N_v, x_v \rangle = a_{12} F + a_{22} G$$

In matrix language, we have

this is almost dN_p in the x_u, x_v basis

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

<u>this</u> is the symmetric matrix representing dN_p as a self-adjoint map

Thus

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = -\begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

Notice two things here

1. We've actually taken the transpose of $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ in order to match the last equations.

2. That $(\ )^{-1}$ is a matrix inverse.

Recalling that

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG-F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

(Why? Well, multiply them and check...)

we see that

$$\begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} = \frac{-1}{EG-F^2}\begin{pmatrix} e & f \\ f & g \end{pmatrix}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

so we can compute the a_{ij} directly:

$$a_{11} = \frac{fF - eG}{EG - F^2} \qquad a_{12} = \frac{gF - fG}{EG - F^2}$$

$$a_{21} = \frac{eF - fE}{EG - F^2} \qquad a_{22} = \frac{fF - gE}{EG - F^2}$$

These are called the <u>Weingarten equations</u>.

Now we have two matrices which claim to represent II_p:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} e & f \\ f & g \end{pmatrix}$$

$\uparrow$

represents dN_p as a linear map in the X_u, X_v basis,

represents II_p as a quadratic form w.r.t. to I_p inner product

hard to compute

$\uparrow$

represents II_p as a quadratic form w.r.t. to the standard dot product in $\mathbb{R}^2$

easy to compute

We know (by definition) that

$$K = \det dN_p = a_{11}a_{22} - a_{12}a_{21}.$$

$$H = \frac{1}{2}\operatorname{tr} dN_p = \frac{1}{2}(a_{11} + a_{22}).$$

Confession!

Note that I got something <u>wrong</u> in the 2007 spring lectures on this subject when I said that

$$\det dN_p \text{ was not } a_{11}a_{22} - a_{12}a_{21}.$$

That was silly: det is det, no matter what basis you're in. What I meant to say was

$$\det dN_p \text{ is not } eg - f^2.$$

Similarly, I said

$$\operatorname{tr} dN_p \text{ was not } a_{11} + a_{22}$$

when I meant to say

$$\operatorname{tr} dN_p \text{ is not } e+g.$$

I'm guilty! Very sorry!

From our equation

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = - \begin{pmatrix} e & f \\ f & g \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1}$$

we know

$$K = \det(a_{ij}) = \det \begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix} \cdot \frac{1}{\det \begin{pmatrix} E & F \\ F & G \end{pmatrix}}$$

$$= \frac{eg - f^2}{EG - F^2}$$ ← watch the signs, here! The − sign disappeared into $\det \begin{pmatrix} -e & -f \\ -f & -g \end{pmatrix}$.

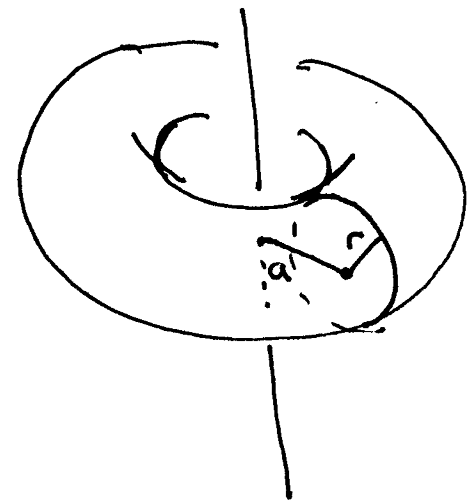
And

$$H = -\frac{1}{2} \operatorname{tr}(a_{ij}) = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2}.$$

$\left(= -\frac{1}{2}(a_{11} + a_{22}), \text{ using Weingarten equations} \right)$

(How cool was that?)

Example.



$$X(u,v) = \Big((a + r\cos u)\cos v,\ (a + r\cos u)\sin v,\ r\sin u \Big)$$

Compute $H(u,v)$ and $K(u,v)$ using e, f, g, E, F, G.

We are going to use a trick:

$$|X_u \times X_v|^2 + \langle X_u, X_v \rangle^2 = \|X_u\|^2 \, \|X_v\|^2$$

so

$$|X_u \times X_v|^2 = E^2 G^2 - F^2$$

or

$$|X_u \times X_v| = \sqrt{EG - F^2}.$$

We observe that

$$X_u = \left(-r\sin u\cos v, -r\sin u\sin v, r\cos u\right)$$

$$X_v = \left(-(a+r\cos u)\sin v, (a+r\cos u)\cos v, 0\right)$$

$$X_{uu} = \left(-r\cos u\cos v, -r\cos u\sin v, -r\sin u\right)$$

$$X_{uv} = \left(r\sin u\sin v, -r\sin u\cos v, 0\right)$$

$$X_{vv} = \left(-(a+r\cos u)\cos v, -(a+r\cos u)\sin v, 0\right).$$

It is easy to compute E, F, G.

$$E = \langle X_u, X_u\rangle = r^2\sin^2 u\cos^2 v + r^2\sin^2 u\sin^2 v + r^2\cos^2 u$$

$$= r^2.$$

$$F = \langle X_u, X_v\rangle = r(a+r\cos u)\sin u\cos v\sin v$$
$$- r(a+r\cos u)\sin u\cos v\sin v = 0.$$

$$G = \langle X_v, X_v\rangle = (a+r\cos u)^2\sin^2 v + (a+r\cos u)\cos^2 v$$
$$= (a+r\cos u)^2.$$

But how best to compute e, f, g?
We invoke the <u>alternate</u> forms

$$e = \langle N, x_{uu} \rangle$$

$$= \left\langle \frac{x_u \times x_v}{|x_u \times x_v|}, x_{uu} \right\rangle = \frac{1}{\sqrt{EG - F^2}} (x_u, x_v, x_{uu}).$$

where (a, b, c) is the "triple product"

$$a \times b \cdot c = b \times c \cdot a = c \times a \cdot b$$

$$= \det \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ a & b & c \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

$$f = \langle N, x_{uv} \rangle = \frac{1}{\sqrt{EG - F^2}} (x_u, x_v, x_{uv}).$$

$$g = \langle N, x_{vv} \rangle = \frac{1}{\sqrt{EG - F^2}} (x_u, x_v, x_{vv}).$$

Courageously, we compute

$$X_u \times X_v = \begin{pmatrix} -r(a+r\cos u)\cos u \cos v, \\ -r(a+r\cos u)\cos u \sin v, \\ -r(a+r\cos u)\sin u \cos^2 v \\ -r(a+r\cos u)\sin u \sin^2 v \end{pmatrix}$$

$$= -r(a+r\cos u)\left(\cos u \cos v, \cos u \sin v, \sin u\right).$$

And then we work out

$$(X_u, X_v, X_{uu}) = -r(a+r\cos u)\left[-r\cos^2 u \cos^2 v - r\cos^2 u \sin^2 v - r\sin^2 u\right]$$

$$= r^2(a+r\cos u)$$

$$(X_u, X_v, X_{uv}) = -r(a+r\cos u)\left[\begin{matrix} r\cos u \sin u \sin v \cos v \\ -r\sin u \cos u \cos v \sin v + 0 \end{matrix}\right]$$

$$= 0.$$

$$(X_u, X_v, X_{vv}) = -r(a+r\cos u)\left[\begin{matrix} -(a+r\cos u)\cos u \cos^2 v \\ -(a+r\cos u)\cos u \sin^2 v + 0 \end{matrix}\right]$$

$$= r(a+r\cos u)^2 \cos u.$$

So we have

$$\sqrt{EG-F^2} = \sqrt{r^2(a+r\cos u)^2} = r(a+r\cos u),$$

and

$$e = r$$

$$f = 0$$

$$g = (a+r\cos u)\cos u$$

This means that

$$K = \frac{eg-f^2}{EG-F^2} = \frac{r(a+r\cos u)\cos u}{r^2(a+r\cos u)^2} = \frac{\cos u}{r(a+r\cos u)}$$

$$H = \frac{1}{2}\,\frac{eG-2fF+gE}{EG-F^2} = \frac{r(a+r\cos u)^2 + (a+r\cos u)^{3}\cos u\, r^2}{2\; r^2(a+r\cos u)^2}$$

$$= \frac{1}{2}\left(a+r\cos u + r\cos u\right)$$

$$= \frac{1}{2}(a + 2r\cos u).$$

Where are the elliptic, parabolic, and hyperbolic points on the torus?

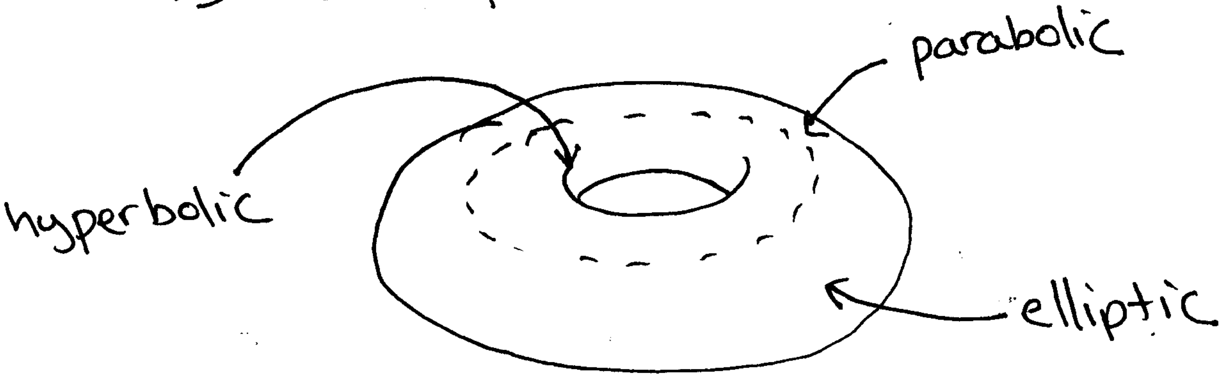
Since $a > r > 0$, $a + r\cos u > 0$. So

$$K < 0 \iff \cos u < 0 \iff u \in (\pi/2, 3\pi/2)$$

$$K > 0 \iff u \in (0, \pi/2) \text{ or } (3\pi/2, 2\pi)$$

$$K = 0 \iff u = \pi/2 \text{ or } 3\pi/2.$$

This corresponds with our intuition



Notice that there seem to be "as many" elliptic points as hyperbolic points....
(coincidence? I think, not.)