The meaning of II_p, continued

A point p on S is called

elliptic if $\det dN_p = K > 0$

hyperbolic if $\det dN_p = K < 0$

parabolic if $\det dN_p = K = 0$, but $dN_p \neq 0$

planar if $dN_p = 0$.

Examples. We can compute these for a few surfaces, where we know the eigenvectors/values of dN_p.

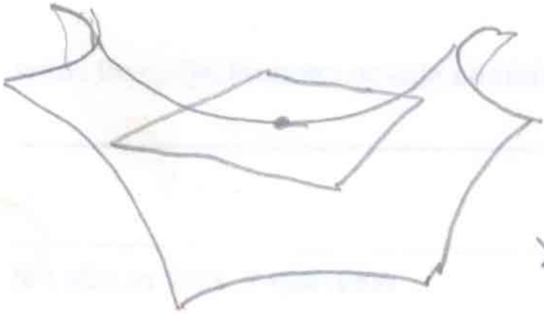
round sphere

dN_p = identity. matrix

$k_1 = k_2 = -1$

$k_1 \cdot k_2 = 1 = K$, $H = 1$

elliptic



hypar

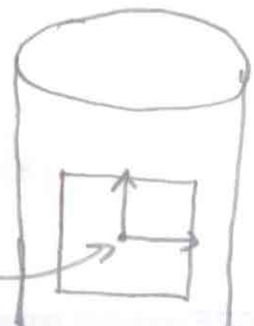
$x(u,v)=(u,v,v^2-u^2)$

$dN_p(a,b,0)=(2a,-2b,0)$

$k_1=-2,\ k_2=+2$

$K=-4,\ H=0$

hyperbolic!



(1,0,0)

cylinder

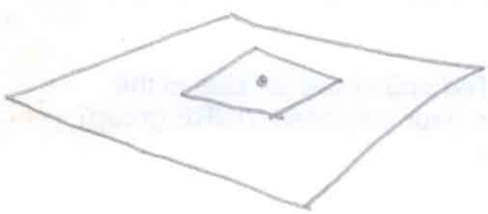
$x(z,\theta)=(\cos\theta,\sin\theta,z)$

$dN_p(a,0,b)=(a,0,0)$.

$k_1=-1,\ k_2=0$

$K=0,\ H=-\frac{1}{2}$

parabolic



plane

$dN_p=0$

$K=0,\ H=0$

planar

We will see later that surfaces with $H=0$ are called <u>minimal</u> surfaces - they are critical points for area.

Neat fact.

Definition. If $K_1 = K_2$, we call this an umbilic point.

Proposition If S is connected and umbilic everywhere then S is part of a plane or sphere.

Proof. Choose local coordinates $x(u,v)$. We know if $w = a_1 x_u + a_2 x_v$, then

$$dN_p(w) = \lambda(p)\, w, \qquad (*)$$

where λ is differentiable (the surface is smooth).

Claim: $\lambda(p)$ is constant.

Using our formula for dN_p, we see that $(*)$ is

$$a_1 N_u + a_2 N_v = \lambda a_1 x_u + \lambda a_2 x_v.$$

for any a_1, a_2. So

$$N_u = \lambda x_u, \quad \text{and} \quad N_v = \lambda x_v.$$

or

$$N_{uv} = \lambda_v x_u + \lambda x_{uv}$$
$$N_{vu} = \lambda_u x_v + \lambda x_{vu}$$

Thus

$$\lambda_u x_v - \lambda_v x_u = 0.$$

But x_u, x_v are linearly <u>independent</u> so this implies $\lambda_u = \lambda_v = 0$.

Since S is connected, this $\Rightarrow$ λ is constant everywhere.

Now if $\lambda \equiv 0$, then $dN \equiv 0$ so N is constant. Thus if we take a point $x(u,v)$ on S,

$$\langle x(u,v), N \rangle_u = \langle x_u, N \rangle + \langle x, N_u \rangle$$

$$= 0 + 0 = 0$$

(and the same for v) so

$\langle x, N \rangle$ is constant, so x is on a plane

Suppose $\lambda \neq 0$. Let

$$y(u,v) = \left(x(u,v) - \frac{1}{\lambda} N(u,v)\right).$$

Then

$$y_u = x_u - \frac{1}{\lambda} N_u = 0,$$

$$y_v = x_v - \frac{1}{\lambda} N_v = 0,$$

so y is constant. Now

$$|x(u,v) - y|^2 = \left|\frac{1}{\lambda} N(u,v)\right|^2 = \frac{1}{\lambda^2}$$

so $x(u,v)$ is on the sphere of radius $1/\lambda$ centered at y, as claimed!

Definition. Given p in S, an <u>asymptotic direction</u> of S is one with normal curvature 0. An <u>asymptotic curve</u> has all tangents in asympotic directions.

Note that ~~only~~ elliptic points have no asymptotic directions. We claim that hyperbolic points have two.

Definition. The Dupin indicatrix at p is the set of vectors in T_pS with

$$\mathrm{II}_p(v) = \pm 1.$$

Computation. We know that if v_1, v_2 are the eigenvectors of dN_p then

$$\mathrm{II}_p(x v_1 + y v_2) = \langle K_1 x v_1 + K_2 y v_2, x_1 v_1 + x_2 v_2 \rangle_{\mathrm{I}_p}$$

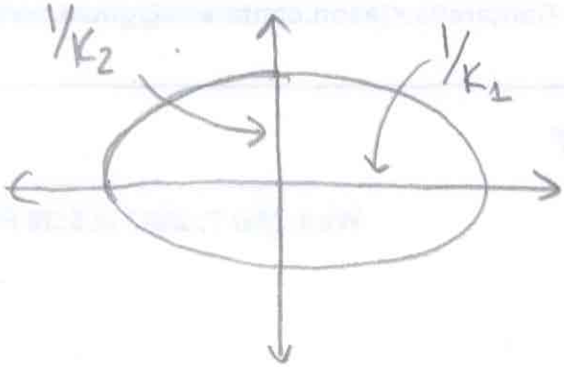
$$= K_1 x^2 + K_2 y^2$$

(this is called Euler's formula)

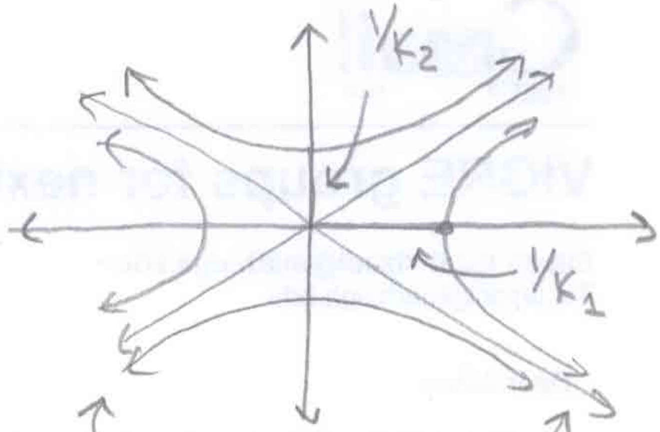
The solutions of

$$K_1 x^2 + K_2 y^2 = \pm 1$$

are either

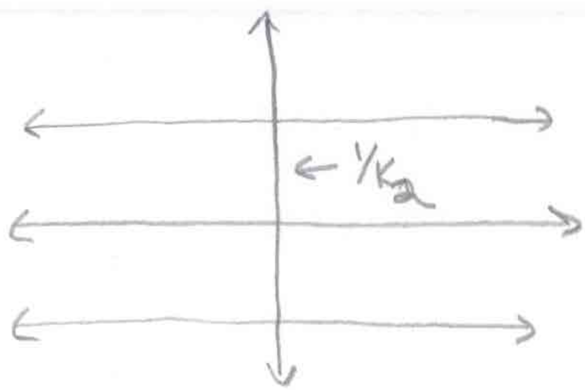


"elliptic"



$y/x = \pm\sqrt{-K_1/K_2}$

"hyperbolic"



"parabolic"
(sorry)

We last define

Definition. A pair of vectors v, w are conjugate if

$$\mathrm{II}_p(v,w) = 0$$

Two directions are conjugate if any pair of vectors parallel to these directions are conjugate.

Examples.

An asymptotic direction is conjugate to itself.

A pair of principal directions v_1, v_2 are conjugate to each other.

At an umbilic point, $\langle v,w\rangle_{I_p} = 0 \iff v,w$ are conjugate.