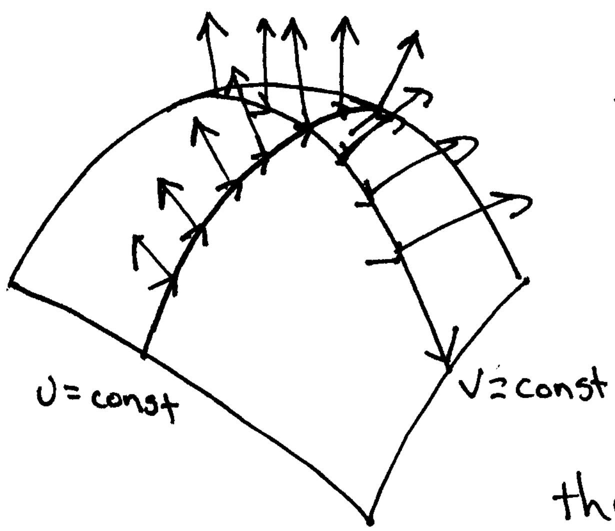
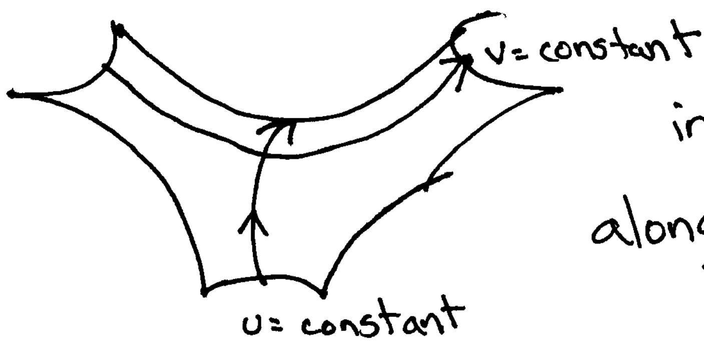
The Gauss Map and the Second Fundamental Form: measuring the curvature of a surface.

We are interested now in describing the geometry of a surface in a more compact way. Here's the idea:



The surface bends the same way along each coordinate line. We can track that bending by following the change in the normal vector to the surface. We want to distinguish this from, where the surface bends in opposite directions along each coord. line.



<u>Definition</u>. A surface S is <u>orientable</u> if there is a differential map $N: S \to S^2$ so that

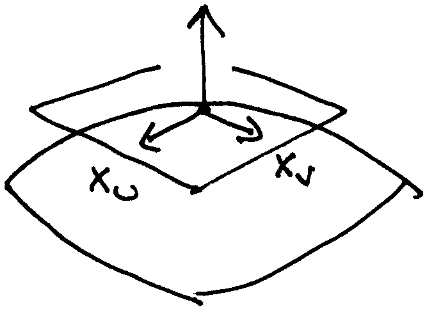
$$N(p) \perp T_p S$$

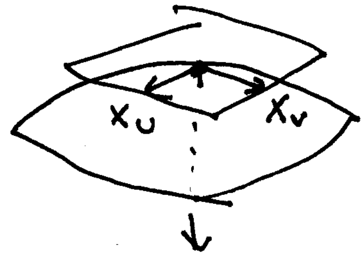
at every point on S.

The map $N(p)$ is called the <u>Gauss map</u>.

We can give the Gauss map explictly in local coordinates $\vec{X}: U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Since $T_p S$ has the basis $\vec{X}_u, \vec{X}_v$ we know that

$$N(p) = \pm \frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$$



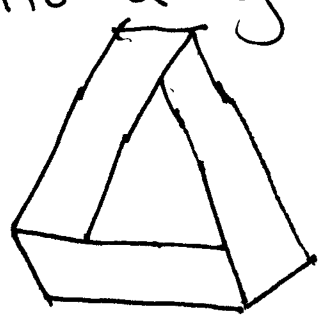


At any given point, there are two choices for the normal vector, so

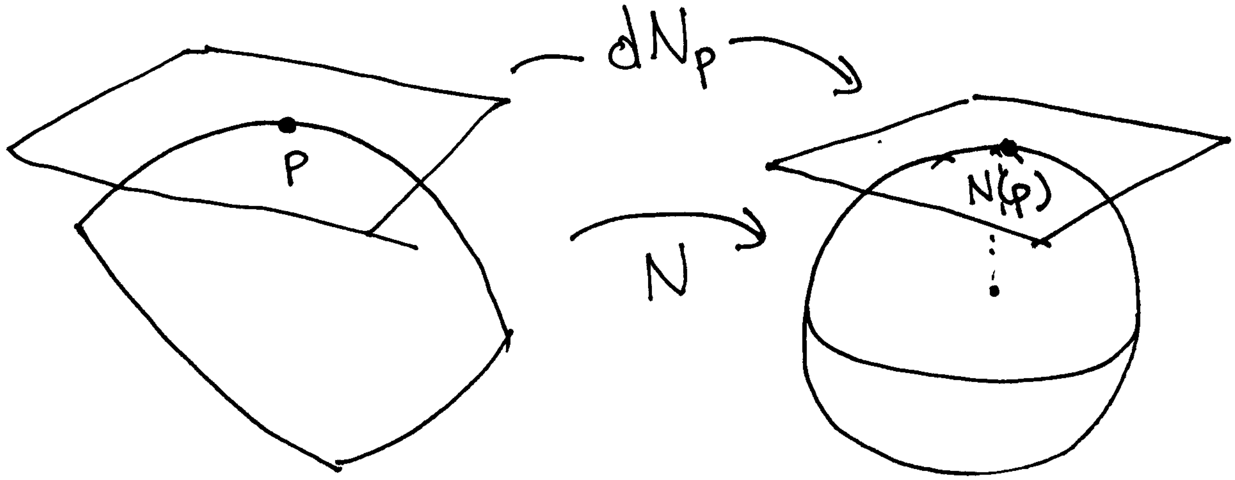
$$\frac{\vec{X}_u \times \vec{X}_v}{|\vec{X}_u \times \vec{X}_v|}$$ agrees or disagrees with N.

If this is -N, we can "fix things" by changing the names of u and v.

$\boxed{\text{Beware.}}$ Not every regular surface <u>is</u> an orientable surface. We might not be able to patch all the local normal fields into a global one. A good example is the Möbius strip at left.



Since we have $N: S \to S^2$,



we have

$$dN_p : T_p S \to T_{N(p)} S^2$$

But $T_{N(p)}S^2$ is normal to $N(p)$, since it's a tangent plane to the sphere, and T_pS is normal to $N(p)$, by definition. So there is a natural map

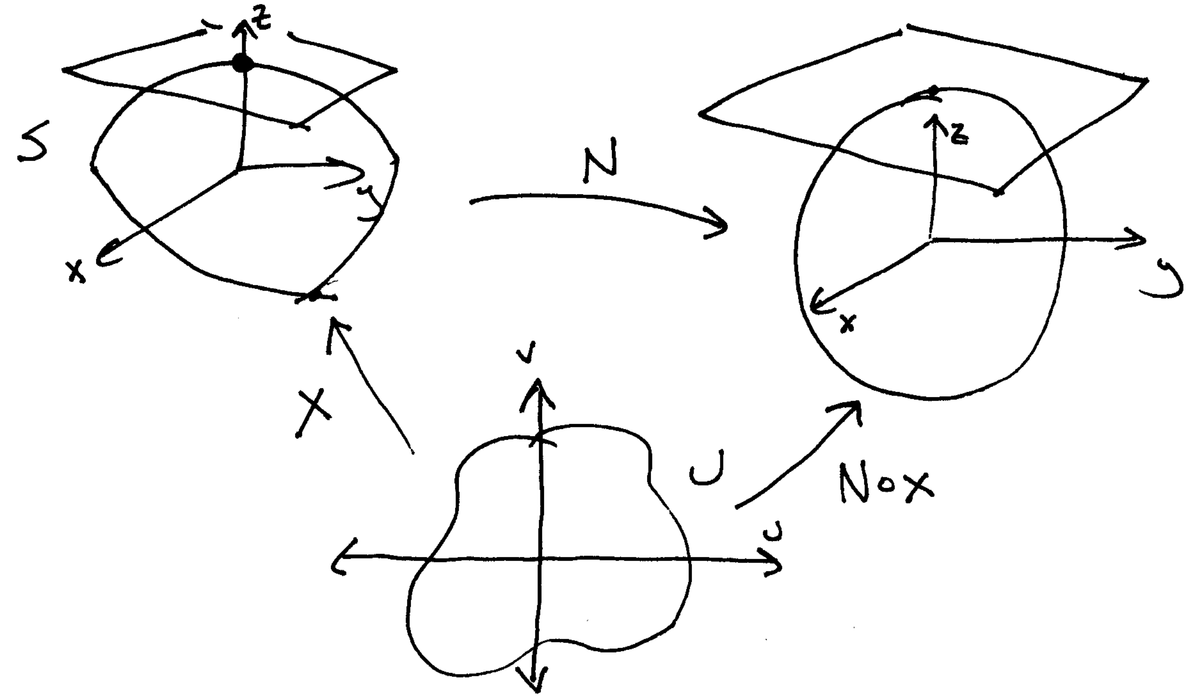
$$T_{N(p)}S^2 \xrightarrow{Id} T_p S$$

and we can think of dN_p as a map

$$dN_p : T_p S \to T_p S.$$

~~Examples.~~

Computing dN_p in local coordinates, and in terms of a consistent system of coordinates on $\mathbb{R}^3$.



Trick: Use $N \circ X$ to define local coordinates on S^2. Then the map N is just

$$N(u,v) = (u,v).$$

So $dN_p = I$. But on $T_{N(p)}S^2$, we are in the basis N_u, N_v and on T_pS we are in the basis X_u, X_v.

So ~~in~~ ~~X_u, X_v~~ on T_pS, we have

$$dN_p : T_pS \to T_pS \text{ given by } \begin{array}{l} X_u \to N_u. \\ X_v \to N_v. \end{array}$$

Examples.

1. The x-y plane.

$$X(u,v) = (u, v, 0). \qquad N(u,v) = (0,0,1).$$

$$X_u = (1,0,0) \qquad N_u = (0,0,0)$$

$$X_v = (0,1,0) \qquad N_v = (0,0,0)$$

so

$$dN_p = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{everywhere.}$$

2. The unit sphere.

$$X(u,v) = (u, v, \sqrt{1-u^2-v^2}) \qquad N \cancel{= \left(\frac{+2u}{\sqrt{1-u^2-v^2}}, \frac{2v}{\sqrt{1-u^2-v^2}} \right.}, 1)$$

$$X_u = \left(1, 0, \frac{-\cancel{2}u}{\sqrt{1-u^2-v^2}}\right) \qquad \cancel{N_u = \left(\frac{2}{\sqrt{1-u^2-v^2}} + 2u\left(-\frac{1}{2}\frac{2u}{(\sqrt{1-u^2-v^2})^3}\right), \right.}$$

$$X_v = \left(0, 1, \frac{-\cancel{2}v}{\sqrt{1-u^2-v^2}}\right) \qquad \cancel{\left. 2v\left(-\frac{1}{2}\frac{2u}{(\sqrt{1-u^2-v^2})^3}\right), 0 \right)}$$

We then have

$$X_u \times X_v = \left(\frac{u}{\sqrt{1-u^2-v^2}}, \frac{v}{\sqrt{1-u^2-v^2}}, 1 \right).$$

$$= \frac{1}{\sqrt{1-u^2-v^2}} \left(u, v, \sqrt{1-u^2-v^2} \right)$$

so

$$N = \left(u, v, \sqrt{1-u^2-v^2} \right) = X$$

and

$$N_u = X_u$$

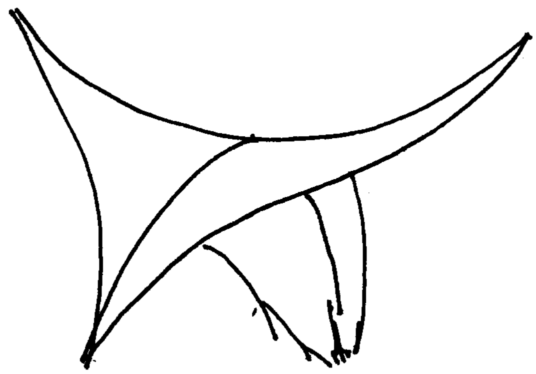
$$N_v = X_v$$

so we have

$$dN_p : T_pS \to T_pS$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

3 The Hyperbolic Paraboloid



$$\vec{X}(u,v) = (u, v, v^2 - u^2).$$

We compute

$$X_u = (1, 0, -2u)$$

$$X_v = (0, 1, 2v)$$

$$X_u \times X_v = (2u, -2v, 1)$$

$$|X_u \times X_v| = \sqrt{1 + 4u^2 + 4v^2} = 2\sqrt{1/4 + u^2 + v^2},$$

so

$$N = \left(\frac{u}{\sqrt{u^2+v^2+1/4}}, \frac{-v}{\sqrt{u^2+v^2+1/4}}, \frac{1}{\sqrt{u^2+v^2+1/4}} \right)$$

and we have $N = \frac{1}{\sqrt{u^2+v^2+1/4}}(u, -v, 1)$.

$$N_u = \left\{ -\frac{1}{\cancel{2}} \frac{\cancel{2}u}{(\sqrt{u^2+v^2+1/4})^3}(u, -v, 1) + \frac{1}{\sqrt{u^2+v^2+1/4}}(1, 0, 0) \right.$$

or

$$N_u = \frac{1}{\sqrt{u^2+v^2+1/4}}\left(1 - \frac{u^2}{u^2+v^2+1/4},\ \frac{+uv}{u^2+v^2+1/4},\ \frac{-\cancel{4}u}{2(u^2+v^2+1/4)}\right)$$

$$N_v = \frac{1}{\sqrt{u^2+v^2+1/4}}\left(\frac{-uv}{u^2+v^2+1/4},\ -1 - \frac{v^2}{u^2+v^2+1/4},\ \frac{-\cancel{4}v}{2(u^2+v^2+1/4)}\right)$$

At $u=v=0$, we have

$$X_u = (1,0,0) \qquad N_u = (1,0,0) = X_u$$

$$X_v = (0,1,0) \qquad N_v = (0,-1,0) = -X_v$$

Therefore, we have

$$dN_p = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

This one requires a little consideration. The columns of this matrix are N_u, N_v, written in the X_u, X_v basis as

$$N_u = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } N_v = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

We notice that in all 3 cases, dN_p was a symmetric matrix. That's not a coincidence!

Proposition. For any vectors $\vec{v}, \vec{w} \in T_pS$, we have $\langle \vec{v}, dN_p(\vec{w}) \rangle_p = \langle dN_p(\vec{v}), \vec{w} \rangle_p$, (where the inner product is I_p: the inner product on $\mathbb{R}^3$).

Proof. Given a basis X_u, X_v for T_pS, we only have to check

$$\langle \vec{X}_u, dN_p(\vec{X}_v) \rangle \overset{?}{=} \langle dN_p(\vec{X}_u), \vec{X}_v \rangle$$

as vectors in $\mathbb{R}^3$. Now this is really

$$\langle \vec{X}_u, \vec{N}_v \rangle \overset{?}{=} \langle \vec{N}_u, \vec{X}_v \rangle.$$

We know $\langle \vec{N}, \vec{X}_u \rangle \equiv 0$, so

$$\frac{\partial}{\partial v} \langle \vec{N}, \vec{X}_u \rangle = \langle \vec{N}_v, \vec{X}_u \rangle + \langle \vec{N}, \vec{X}_{uv} \rangle = 0.$$

and similarly

$$\frac{\partial}{\partial u} \langle \vec{N}, \vec{X}_v \rangle = \langle \vec{N}_u, \vec{X}_v \rangle + \langle \vec{N}, \vec{X}_{vu} \rangle = 0.$$

Thus we have

$$\langle \vec{N}_v, \vec{X}_u \rangle = -\langle \vec{N}, \vec{X}_{uv} \rangle = \langle \vec{N}_u, \vec{X}_v \rangle$$

as desired.

Comment. Does this mean that our previous matrix, dN_p, is symmetric? No!

Suppose

$$N_u = a X_u + c X_v$$

$$N_v = b X_u + d X_v$$

so

$$dN_p = \begin{bmatrix} a & b \\ \cancel{b}c & d \end{bmatrix}.$$

If I_p is given by the matrix

$$I_p = \begin{bmatrix} \langle X_u, X_u \rangle & \langle X_v, X_u \rangle \\ \langle X_u, X_v \rangle & \langle X_v, X_v \rangle \end{bmatrix}$$

then this means that the matrix

$$I_p \, dN_p = \begin{bmatrix} \cdot & b\langle X_u, X_u\rangle + d\langle X_v, X_u\rangle \\ a\langle X_u, X_v\rangle + \cancel{d}\, c\langle X_v, X_v\rangle & \cdot \end{bmatrix}$$

is symmetric. Or that

$$a\langle X_u, X_v\rangle + c\langle X_v, X_v\rangle = \langle aX_u + cX_v, X_v\rangle$$
$$= \langle N_u, X_v\rangle$$

$$\parallel$$

$$b\langle X_u, X_u\rangle + d\langle X_v, X_u\rangle = \langle bX_u + dX_v, X_u\rangle$$
$$= \langle N_v, X_u\rangle,$$

which is exactly what we proved above.

This leads us to define

Definition. The quadratic form

$$\mathrm{II}_p(\vec{v},\vec{w}) = -\langle dN_p(\vec{v}), \vec{w}\rangle_p$$

is called the <u>second</u> <u>fundamental</u> <u>form</u> of the surface S at p.